The Austrian Theory of the Natural Rate of Interest: A Neoclassical Critique, Part Three

Pavel Potužák, Department of Economics

4 NATURAL RATE OF INTEREST UNDER VARIOUS THEORETICAL ENVIROMENTS

In this part, we will relax the assumption of constant marginal productivity of capital. The exogenous flow of real income will be introduced as well. As we will see, much richer conclusions might be drawn by these two extensions. However, all will demonstrate again that the PTPT is just a special case of a more general theory.

In this section, we will keep using the two-period model. This decision has one big advantage and one big disadvantage. The good thing is that all conclusions from this section can be directly compared with those from previous sections. However, one important aspect of the Austrian theory of capital and interest will be lost.

The core of the Austrian capital theory is that roundabout methods of production are more productive.[1] In other words, (wisely chosen) time extension of the production process should lead to a higher output. However, the increments in the output are not proportional, because it is generally believed that this process exhibits decreasing marginal (physical) productivity. The presence of time preference together with decreasing marginal productivity will eventually bring the process of the time extension of production to a halt. Although an infinite time extension of production could possibly create an infinite output, this option is never chosen by rational agents. Never-ending postponement of present consumption is impossible due to time preference.[2]

However, our two-period model does not allow us to analyse this aspect of the theory of capital and interest. With regard to time dimension, only two options are available for the investment of the factors of production. They can either be invested in processes (Hayek PTC) that provide consumption goods immediately or in a relatively short period of time (i.e. in period 0), or they can be invested in longer processes that will take one period. The longer processes will then create consumption goods in period 1. Consistently with the assumption stated above, the given number of factors of production will produce more if they are invested for one period compared with their immediate use. At the same time, the longer the time for which the factors of production are tied up, the higher the eventual output, even though the marginal increments gradually diminish. However, in the two-period model, the number of periods cannot be extended. In other words, capital cannot be enlarged in height (Wicksell:LPE). As a result, in the two-period model decreasing marginal productivity of capital cannot be reflected in the time dimension.

Nonetheless, we would like to introduce some kind of diminishing marginal productivity of capital even in the two-period model. Thus, suppose that every additional unit of input invested in a longer process (i.e. in the process that takes one period) increases future output, but at a decreasing rate. As more factors of production are directed to the longer process, the

[1] Bohm-Bawerk associated the productivity of roundabout methods with his third ground for interest. Mises (HA:???) preferred term "longer methods" rather than "roundabout methods", because … !!!

[2] Here we should add time preference in the first sense, because the huge under-provision in the present, thus almost infinite marginal utility from present consumption and the resulting enormous size of the MRS (i.e. time preference in the first sense) will not allow never-ending postponement of present consumption and indefinite lengthening of the production process.

output of consumption goods in the next period increases. Yet, these increments are still lower and lower. As a result, in the two-period model decreasing marginal productivity can be reflected only in the breadth dimension of capital.

Obviously, with more units of input invested in a longer process, the average investment time of the entire stock of inputs (the average period of production in the Bohm-Bawerkian system) increases (Hayek PTC, …). Nevertheless, the maximum time for which one single unit of input can be invested is just one period. Longer time extension is not possible.

Greater fruitfulness of longer methods and their diminishing marginal productivity can be illustrated in a simple Fisherian diagram (Figure no. 20). One possible interpretation of this scheme is as follows: Point A can be considered as the initial endowment of present goods that might be transformed into future goods if properly invested. The first forgone present good might produce 5 future goods. However, this physical productivity gradually falls, since the second present good may produce just 3 future goods etc. We can imagine an interval (from point D to the left), in which the production process is so technically inferior and unfortunate that the marginal output of future goods falls short of the number of marginal present goods invested. In this case, the slope of the investment opportunity line is lower than one (in absolute value).

However, our exposition would be more in line with the Austrian theory of capital if we interpreted point A as follows: This point represents the maximum amount of present consumption goods that can be produced if all factors of production are used only in direct methods of production.[3] The investment of factors of production in a longer process decreases present output, however, the decline in present output is more than compensated by an increase in output of future consumption goods (point B). The fruitfulness of this reallocation of resources is, nevertheless, limited since the marginal increments of future output gradually fall. Illustration of the varying time extension of production would require (at least) a three-dimensional graph. Yet, this will not be used in our analysis.[4]

Even this simplified version can provide us with key insights. As has been demonstrated before, the subjective exchange ratio between present goods and future goods depend on the MRS, i.e. on the time preference in sense one. However, MRS is an endogenous concept, because it depends on the stream of income. The indifference curve alone cannot determine the equilibrium interest rate. To close the model, one more curve (or equation) is required. One more relationship is necessary to determine, which particular point on the indifference curve will be chosen. In examples with shipwrecked sailors, the system was closed owing to the assumption of constant productivity. At this place, we assume decreasing marginal productivity of the invested capital.

By varying the amount of inputs invested in longer processes, the intertemporal flow of income might be changed. And because this flow affects the optimum MRS, the (real) natural rate of interest is co-determined by the marginal rate of substitution (subjective element) and the marginal productivity of capital (objective element).

Hayek in his magnum opus on capital (PTC, …) thoroughly explained that the relative impact of time preference and productivity in determining the natural rate of interest depends on the relative curvature of the indifference curve compared with the investment opportunity curve. At this place, we will extend his approach.

---

[3] To be more precise, we have to add that there is no endowment of future factors of production. In other words, future output is zero, unless present factors of production are engaged in longer processes. A picture of an economy, in which there is an endowment of factors of production in the future that might be used also in short processes in the future is presented in Figure no. 33.
[4] See Hayek PTC

Even though this model is quite simple, it enables us to analyze many aspects in the theory of interest. Panel a) on Figure no. 21 portrays a situation in which the subjective discount rate (time preference in sense two) is relatively high and the productivity of inputs invested in a longer process falls quickly. As a result, the slope of the indifference curve at the 45º line is higher than the slope of the opportunity line. The optimum point lies to the right of the diagonal line. Compare this optimum with panel b), representing a relatively patient individual (the subjective discount rate and hence the slope of the indifference curve at the diagonal is quite low) and just slowly decreasing marginal productivity of capital. In panel a, the optimal time shape of consumption is decreasing, whereas in panel b it is increasing. However, as can be seen in the picture, it is quite difficult to say, in which panel the real rate of interest is lower. In the first figure, the interest rate is pushed down by rapidly diminishing marginal productivity, while a relatively high rate of impatience drives it up. In panel b, exactly the opposite statement might be said.

At this point, we can discuss in a more detail why we departed from the Murphy´s approach to time preference in sense one and why we separated the third reason for interest from the previous two. In our reasoning, the first two causes influence the slope of the indifference curve – MRS – the time preference in the first sense; they represent the subjective element in the theory of interest. The third reason is embodied in the investment opportunity line; it stands for the objective or productivity element.[5] All three causes together (the marginal rate of substitution - MRS and the marginal rate of product transformation – MRPT) co-determine the natural rate of interest. Murphy suggested that all three reasons are responsible for the time preference in sense one – the exchange ratio between present and future goods. However, it is more convenient to keep the phenomenon of time preference in sense one just for the subjective part of the model (MRS and the saving function) and separate the third reason for the productivity element (MRPT and the investment function). Time preference in sense one in our reasoning represents the subjective(!) exchange ratio between present and future goods, i.e. any MRS along the entire indifference curve. The eventual objective ratio between present and future goods (but also the equilibrium MRS) then depends also on the productivity element of the model. It is the point where the slope of the indifference curve (determined by the two causes) and the slope of the investment opportunity line (determined by the third cause) coincide. It can be said that Murphy´s reasoning of time preference in sense one is about the point of general equilibrium (one particular and optimal point at the indifference curve). Our reasoning about time preference in sense one contemplates any point on the indifference curve and it seems to be more in line with standard neoclassical reasoning (Becker, Olson …). The third reason is then required to find the eventual equilibrium, i.e. one particular point of optimum at the indifference curve.

The previous reasoning allows us to show a very simple relationship between the Fisherian diagram and the loanable funds market. Both models are crucial in the discussion about the underlying factors of the natural rate of interest. Suppose that the investment opportunity line is close to linear (Figure no. 22, panel a). Consider an increase in the subjective discount rate (from ρ_1 to ρ_2), which might be represented by an increase in the slope of the indifference curve at the diagonal line. The equilibrium of this economy moves from point E_1 to point E_2. Notice that the equilibrium natural rate of interest is not much affected. Slowly decreasing marginal productivity is reflected by a very flat investment curve presented in panel b. The increase in time preference (in sense two) leads to a shift of the saving curve to the left. In the end, the equilibrium quantity of invested capital declines leaving the natural rate of interest

[5] and it determines the investment function

almost unaffected. In this particular situation, the key factor of the natural rate of interest is the physical productivity of capital, not time preference.[6]

Figure no. 23 illustrates the opposite situation. The subjective elasticity of substitution is very high (θ close to 0), which results in almost linear indifference curves. Exogenous increase in marginal productivity of capital (e.g. due to positive technological shock) shifts the opportunity line outwards and the investment curve to the right. The amount of capital invested grows, however, the natural rate of interest is almost the same as before because the saving curve is close to linear. In this particular situation, the natural rate of interest is determined solely by the time preference (in sense two). As we can see again, pure time preference theory is a special case in a more general theory of interest. It is valid just for a very high intertemporal elasticity of substitution in consumption.

If the elasticity of substitution was lower (higher θ), the increase in productivity would have much bigger impact on the interest rate (Figure no. 24). Notice that the indifference curves are much more curved and the saving curve is much less elastic. It is theoretically possible that if the elasticity of substitution is very low ($\theta > 1$) and the preferred path of the consumption stream is much smoothed (E_1 and E_2 are both close to 45º line), the resulting saving curve might be downward sloping. Figure no. 25 illustrates this peculiar situation. In this case, the natural rate of interest is determined rather by the productivity of capital. However, the resulting amount of invested capital paradoxically falls after the increase in productivity.[7]

The craziest situation may occur for an extremely low elasticity of substitution ($\theta >> 1$). A sudden increase in productivity should lead to a fall in the equilibrium interest rate, because the saving curve exhibits not only a decreasing shape, but it is even flatter than the investment curve (Figure no. 26). However, in this case the price mechanism most probably does not work (Wicksell ???), because the natural rate of interest has a tendency to increase after the shock rather than fall down. The reason lies in the fact that investment (demand) exceeds saving (supply). As a result, the initial imbalance will expand over time and the equilibrium lower interest rate can never be achieved. In more complicated dynamic models (e.g Diamond model), such a low elasticity of substitution may lead to sunspot equilibria and self-fulfilling prophecies.

Furthermore, Fisher (1930:???) envisioned a situation in which the simultaneous existence of very patient people and very poor investment opportunities resulted in the negative natural rate of interest. Figure no. 27 clearly shows that only interest rate less than zero equilibrates investment and saving in this case. However, it must be also assumed that it is impossible to store present goods to the future, so the linear part with slope one (dashed line from point B in panel a) is not effective and the actual resource constraint is thus represented by the entire concave opportunity curve.

Notice that the natural rate of interest might be negative even if the economy is populated by "Misesian" people who prefer the given goal to be achieved as soon as possible ($\rho > 0$). As can

---

[6] Hayek(???Time preference and productivity: A Reconsideration) in his later paper on capital theory predicted that the sudden (unexpected) decrease in saving may result in a very unfortunate interruption of the creation of capital structures. This is then reflected by a highly curved opportunity line, where the restructuring of the process of production from longer methods to shorter methods requires a very high sacrifice of future output in order to obtain one unit of present output. Such an abrupt change in time preference has similar consequences as a halt of the monetary expansion at the very peak of the boom. More on this is presented in Potužák (Chapter 2). As a result, an unexpected fall in saving leads to the fact that the natural rate of interest is determined mainly by the time preference, provided that it is difficult for the production process to reallocate resources from longer methods to shorter methods.

[7] Investment and saving curves should not be linear. Yet, the linear shape is constructed just for simplicity and as an approximation around the given equilibrium.

be seen, this a-priori positive time preference in sense two is represented by the slope of the indifference curve exceeding one at the 45° line. However, time preference in sense one ε = MRS – 1 is negative and along with the diminishing marginal productivity of capital they co-determine the (negative in this case) natural rate of interest. Once again, we arrived at a theoretical possibility that is unthinkable in the pure time preference theory.

At the same time, it must be clearly understood that the zero-bound on the interest rate is not binding in this economy (as might be suggested by the FG distance on the horizontal line in panel b). The zero bound is a problem of the nominal rate of interest, not the real rate of interest studied here. As can be seen in panel a), the present output of consumption goods is larger than the future output, therefore the negative real rate of natural interest might be easily generated by a high rate of price inflation (see equations 24 and 26) if this rate exceeds the given positive (or zero) level of the nominal rate of interest.

So far, we assumed that the economy is populated by identical agents. Another interpretation could be that we displayed the situation of a typical (or average) consumer. Thus, we were not concerned with the possibility that at the individual level saving need not equal investment. In other words, tangents to the indifference curve and to the opportunity line at the given market rate of interest were found at the same point.

However, if agents were not identical, the tangencies for the given market rate of interest might be posited at different points. Consider a situation of one particular individual in Figure no. 28. If the (income) endowment of this individual was at point A,[8] and no investment opportunities were available, for the given market rate of interest r_E, the optimum of this consumer would be at point E_0. This consumer would be a saver with the optimum amount of saving DA. Yet, if we allow him to engage in investment activity, he might use the opportunity of a higher physical return on every dose of investment that exceeds the market real rate of interest r_E. The investment activity is profitable, until the marginal rate of return (the slope of the investment opportunity line minus one) is greater than the real rate of interest r_E. He is motivated to increase the investment activity as long as this condition is met. The optimum point is at F, where the real rate of return is equal to the real rate of interest. At this point, the present value of his income stream is maximised (Fisher 1930:???) and the difference between his returns (BF) over costs of investment (original investment plus interest, i.e. BG) is the greatest possible (i.e. FG) (Stigler The theory of price???:316). As can be seen, the optimum amount of investment is AB.

With regard to consumption, higher income allows him to consume more in both periods compared with the original situation. His new optimum is at point E_1. He is definitely better off, as the new optimum lies at a higher indifference curve. His present consumption rises from 0D to 0C, so his saving is reduced from AD to AC. Now, saving is too low to finance his optimum investment AB. Hence, he becomes a debtor - his optimum borrowing is represented by AB-AC=BC.

If all agents were like this one, the presented situation would not be sustainable, since it is impossible for everyone to be a debtor. The equilibrium market real rate of interest must go up to equalise saving and investment. Only if agents were heterogeneous, the presented optimum would be stable. However, this would require that there were agents with the excess of saving over investment. These agents must be more patient and(or) they must have less favourable range of investment opportunities compared with the agent considered here. Nevertheless, further discussion about the heterogeneity of agents will be postponed to the next section.

---

[8] In the next section, we will relax the assumption of zero future (income) endowment, hence we will allow any shape of the income stream, not only (A,0).

So far, we demonstrated that the natural rate of interest is determined by the time preference and the marginal productivity of capital. Yet, Olson (???) listed the following set of determinants of the interest rate…. The last part of this section deals with an exogenously given flow of real income. + Fisher (1907:150)

However, before we discuss the impact of an exogenous flow of real income on the interest rate, let us discuss objections of the PTPT authors against the conclusion presented above, namely that the increase in productivity should raise the natural rate of interest. The strongest opposition against this prediction can be found in Rothbard (???). In this book, Rothbard has extended the original Mises´s objection. If the invention (unexpectedly) increases productivity of capital goods and hence the resulting output of consumption goods, the impact on the interest rate is only temporary. According to Rothbard, higher revenues from the extra output should only lead to temporary profits for the users of capital. Sooner or later, the expanded output of consumption goods must reduce prices of these goods and/or higher profitability of capital goods must be reflected and imputed in higher prices of capital goods.[9] This process will continue till the point, in which no profits are left to be reaped. At this point, the difference in value between output and inputs falls back to the level dictated by the pure time preference.

Our response to Rothbard´s objection must be separated into several parts. First, if the growth in productivity of capital arises in the economy with constant returns to capital, the real rate of interest can never fall back. Suppose, for example, that in the rice economy the net return to capital suddenly rises from 10 % to 15 %. It is not necessary to repeat the arguments from the previous section that there can be no equilibrium real rate of interest other than 15 %. In the loanable funds model, constant marginal productivity is reflected by a horizontal investment curve that solely determines the real rate of interest.[10] After the exogenous productivity shock it shifts upwards to a new level of 15 % (see Figure no. Rothbard 1).

Furthermore, Rothbard´s argument deals with the behaviour of the value difference between output and invested inputs. As we have already seen, this value difference is associated with nominal interest and says nothing about the market exchange ratio between present goods and future goods. Increase in productivity in the rice economy should move the optimum exchange ratio between present rice and future rice to 1.15. However, nominal rate of interest might reach any size depending on the value of ρ and θ.

Recall again examples in Figure no. 19 and equation (28). Assuming r = 15%, ρ = 0 % and θ=1, the nominal rate of interest in panel a) stays the same at the level of 0 %. However, output growth will increase to 15 % and prices will fall by 15 %. Similarly, in panel c (ρ = 4 %) the value difference between present and future rice will remain at the same level of $80, so the nominal rate of interest will be 4 % as before. As can be seen, Rothbard´s prediction about the return of the value difference back to the level dictated by the pure time preference (ρ) holds at least for θ=1. Yet, the real rate of interest is undoubtedly affected by the increase in productivity. For the given rate of nominal interest, higher real interest rate is ensured by a sharper decline in prices.

As we already know, the equality between the nominal interest rate and the pure time preference ρ was valid only for unitary intertemporal elasticity of substitution. Yet, the nominal interest rate in panels b) and d) in Figure no. 19 does not fully reflect the subjective discount rate. A rise in productivity in panel b) will result not only in a permanent increase in the real rate of interest to 15 %, but the nominal interest rate should grow to 7.24 % from 4.88

[9] However, similar reasoning might be found also in Fisher (1913, 1907).
[10] This determination of the real rate of interest was especially emphasized by F. Knight (On Mises, …)

%.[11] The additional decline in prices will be only 2.36 %, since output will grow over time by only this extra percentage amount. Thus, after the increase in productivity, investment in present rice for \$2,000 will result in the net nominal interest of 7.24 % x \$2,000 = \$145, which is more than the initial nominal interest of \$100. Since prices in panel b) do not fall fast enough, increase in physical productivity may result also in the growth in value productivity. A similar result is obtained in panel d), in which the nominal rate of interest increases from 7 % to 9.36 %.

Hence, according to our two period model Rothbard´s prediction about the return of the rate of interest back to its initial level is not correct. With constant MPK, the increase in productivity will affect not only the real rate of interest, but (except for $\theta=1$) it will influence also the nominal interest rate, i.e. the difference between the value of expended inputs and the value of the resulting output.

A similar outcome will be reached, if we relax the assumption of constant MPK. Figures no. 23 – 26, which are built on diminishing MPK, clearly show that the real rate of interest must be affected by higher productivity. The impact on the nominal rate of interest then depends on other parameters of the model (e.g. θ). However, a precise return to exactly the same level of nominal interest is not very plausible.

Nevertheless, Rothbard´s reasoning seems to be directed at a more dynamic environment than analysed here with the help of a simple two-period model. Thus, if this basic model (of diminishing MPK) is extended to more periods (even to infinity in the limiting case), different outcomes might be reached compared with a simple two-period model. As will be seen in section 5.1, in the infinite horizon model the impact of productivity on the real (and nominal) rate of interest critically depends on the permanence and nature of the productivity shock. The return of the interest rate back is possible, although the mechanism is different from Rothbard´s reasoning, and this return will certainly take much more time.

At the end of this section, let us mention opinions of other PTPT authors about the impact of the productivity growth on the rate of interest. Frank Fetter (???) thought that the productivity growth leads to a better provision of present goods. As a result, the interest rate must, according to this early PTPT theorist, fall rather than rise.

The objection against this idea is rather simple. There is no reason to expect that the productivity growth will not affect the provision of future goods also in the positive direction (Pellengahr). Thus, if the relative provision of present and future remains the same, Fetter´s argument is not tenable.

And finally, R. Garrison believed that the productivity growth may decrease time preferences, as it usually brings about higher average income (???:???). This argument was first presented by Fisher (1930). As we will show in section 5.1, this outcome is possible. However, the initial increase in the real rate of interest is inevitable and, moreover, higher real rate of interest should last for a relatively long time.

4.1 EXOGENOUS FLOW OF REAL INCOME, HETEROGENEOUS AGENTS AND THE NATURAL RATE OF INTEREST

The exogenously given flow of income can be understood as an extremely degenerated form of the investment opportunity line (Figure no. 29). Hayek (???:???) argued that this might be the case if all factors of production consist of permanent (and non-renewable) resources. It is assumed that these resources provide a definite flow of goods and services. However, a

[11] Recall that we assume ρ = 0 % and θ=2 in panel b), and ρ = 4 % and θ=2 in panel d).

service at the given point of time cannot be moved to any other period. Since the resources are permanent, factors of production here considered do not represent capital, but only land or labour. To keep the analysis as simple as possible, we will again contemplate the two-period model.

The exogenous flow of income can take any shape. The resulting natural rate of interest might be found either mathematically or graphically. We will start with a graphical approach. Mathematical solution is shown in Appendix 2. Consider an economy populated by people with an identical discount rate, who earn the same real income every period. This situation is closely related to the vision of Mises about an evenly rotating economy, because all processes in the economy are repeated every period in the same way and the same level of income is earned every period. It can be easily shown that the only equilibrium rate of interest is $r_E = \rho$ (Figure 29). Any real interest rate lower than r_E (e.g. r_1) will result in the excess of demand for present goods over their available supply (see $C_0^* - Y$ in panel b). Interest rate higher than r_E will have the opposite effects. Excesses of supply or demand will eventually move the rate of interest back to the level of ρ. Notice that the optimum consumption stream of each individual will be perfectly smoothed (because $r = \rho$, see equation 19 above) regardless of the elasticity of substitution $1/\theta$. Since all agents are identical there will be no individual saving or borrowing, because every agent will exactly consume his or her income endowment in that particular period. As a result, there will be no intertemporal market (recall the discussion with Garrison in section 3), even though the real interest rate must exist to guarantee equilibrium. Any deviation of the real rate of interest from ρ will immediately create this market, however, it will be characterised either by a surplus of present goods or their deficit. Thus, in this case the only general intertemporal equilibrium is zero individual saving, non-existence of the intertemporal market and positive real rate of interest at the level of ρ. The mathematical solution is provided in Appendix 2, section A, equations 18-22.[12]

In this particular economy, the Böhm-Bawerkian first reason does not operate, because the income endowment is the same in both periods. The third reason (higher productivity of roundabout methods) is not effective either, since there is no capital in the economy; it is a pure endowment economy. As we can see, in this economy (and virtually only in this economy), the natural rate of interest is solely determined by the time preference (in sense two). Here, not only the phenomenon of interest as such but also the particular size of the rate of interest exists only due to the fact that people prefer present satisfaction to future satisfaction ($\rho>0$).[13] Only in this economy, the PTPT is correct. Mises had maybe this model of pure constant endowment economy in mind, when he envisioned ERE and the dominance of the time preference in the interest theory.

Now we relax the assumption of a constant flow of income. Misesian economists would argue, that the key assumption of the ERE is then violated. Nevertheless, varying income stream is so pervasive in the real economy that it must be studied here as well. Consider an economy either with an increasing flow of income (Figure no. 30, panel a) or a sharply decreasing flow of income (panel b). In panel a), only a positive natural rate of interest is consistent with the intertemporal equilibrium. Both Bohm-Bawerkian grounds for interest are effective, because people are better provided for in the future and they discount future utilities

---

[12] The solution that $r=\rho$ for constant income flow is valid regardless of the fact whether people have identical size of income or earn different incomes. See Appendix 2, section A and B. Furthermore, since $r=\rho$, consumption will be smoothed for all levels of constant income. As a result, there will be no individual saving either on the part of rich or the poor. Income in every period will be consumed in full regardless of its size.

[13] We can add that if ρ was zero, the natural rate would be zero, if ρ was negative, the natural rate would be negative.

($\rho>0$). The natural rate of interest should be higher than the subjective discount rate (and hence higher than the natural rate of interest shown in panel a) of Figure no. 29) due to the presence of the first ground. Thus, higher marginal valuation of present goods is also supported by their relatively lower present provision. Since $r>\rho$, the optimum time shape of consumption is increasing (see the Euler equation 19 in section 3, or equation 6 in Appendix 2).

On the other hand, in panel b) a negative real rate of natural interest is the only level that will equilibrate demand and supply of present (and future) goods. Panel b) once again represents an economy, where people exhibit time preference in sense two ($\rho>0$), i.e. they prefer the given want to be gratified as soon as possible, but not in sense one ($MRS<1$, $\varepsilon < 0$), i.e. they do not prefer present goods to future goods (exactly the opposite is true). Consequently, the natural rate of interest in this economy is negative. The reason lies in the fact that the first ground for interest works in the opposite direction and it is stronger than the second ground ($\rho>0$). The mathematical solution is provided in Appendix 2, section C, equations 27-31. As can be seen from equation (31), future income must be lower by at least ρ% compared with present income to depress the natural rate of interest below zero. In other words, future income must be sufficiently lower compared with the present income to achieve a premium of future goods over present goods. Very low future income will gratify wants of very high urgency and if the representative present good cannot be moved to the future as we assumed in the beginning, the eagerness to postpone goods (but not the given satisfaction) to the future must decrease the natural rate of interest below zero. Since $r<\rho$, the optimum time shape of consumption is decreasing (see the Euler equation 19 in section 3, or equation 6 in Appendix 2).

If all individuals are identical, individual saving will be zero in both panels of Figure no. 30. If their income streams differ (even though being of the same time shape, i.e. increasing or decreasing) people in one group might become debtors and the other creditors. This conclusion is especially interesting for generally decreasing incomes. Individuals with a very sharp decline in the income stream will become creditors even for a negative real rate of interest (see Figure no. 30Bb). They will lend more present apples in exchange for a lower amount of future apples (the loan of $Y_0^B - C_0^{B*} = C_0^{A*} - Y_0^A$ is lower than the repayment $C_1^{B*} - Y_1^B = Y_1^{A*} - C_1^A$). Mathematical discussion is provided in Appendix 2, section C.

Furthermore, for the given ρ (and θ) in the economy, a flow of income can be exactly determined that will lead to zero natural rate of interest. This particular flow was discussed in section 2.4. In Appendix 2, section C, it is explicitly defined for logarithmic utility function as $Y_1=Y_0/(1+\rho)$. Panel c) in Figure no. 3 illustrates how this income stream might be found. It lies at the perpendicular line to the 45º line that exactly touches the highest possible indifference curve. Obviously, for impatient people ($\rho>0$), this particular income stream must be decreasing ($Y_1<Y_0$). In other words, present must be better provided for than future.

As we can see, even this simple Fisherian model might answer crucial questions in the theory of interest and fundamental questions about the optimum intertemporal consumption behaviour of people. First, it is quite easy to find an equilibrium size of the natural rate of interest. As was demonstrated above, it can be positive, zero or negative regardless of the positivity of the subjective discount rate. Its negative value, i.e. the preference for marginal future goods over the marginal present goods, which is at variance with the pure time preference theory, is caused by a sharply diminishing time shape of the aggregate income in the economy. Hence, if people expect a reduction in their well-being in the future (e.g. due to an expected future stringency of economic conditions at the beginning of a very long recession), natural rate of interest might fall below zero.

Next, the optimum path of consumption can be also easily determined, as it depends on the difference between the real rate of interest and the subjective discount rate. Surprisingly, it does not depend on the particular time shape of income. Thus, at the individual level a decreasing time shape of optimum consumption stream might be consistent with an increasing time shape of income and vice versa. To see this, move the income endowment A^A in Figure 30Ba along the budget line closer to the vertical axis such that this point will eventually lie to the left of the 45º line. In such a case, $Y_1^A/Y_0^A > 1$ but $C_1^{A*}/C_0^{A*} < 1$ (see also Appendix 2, Figures in section 1).

And finally, if flows of income vary across individuals that have an identical subjective discount rate, we may decide whether the particular individual will become a debtor or a creditor for any size of the natural rate of interest in the economy. Debtors (e.g. individuals A in Figure no. 30Ba) are characterized by the fact that the growth rate of their income stream (Y_1^A/Y_0^A-1) is higher than the growth rate of the income stream of creditors Y_1^B/Y_0^B-1 (panel b).

An interesting implication of the last point is that the net borrowing/lending position of each individual does not depend on the absolute average size of income, but rather at its time shape. Thus, individuals with very high present income might be debtors if they expect even higher income in the future. Their high demand for present goods might be satisfied by savings of relatively poor people having low present income who expect its sharp decline over time. Obviously, the number of small savers must be large enough to meet the demand of important borrowers. Furthermore, according to equation (29) in Appendix 2, very high future income of large borrowers will drive up the real interest rate which acts as another brake that will ensure equilibrium in the intertemporal market. The irrelevance of the size of income stems from the fact that all individuals have the same subjective discount rate and that their optimum consumption flow depends only on the difference between the real interest rate and the subjective discount rate.[14] Hence, if the growth rate in income of the individual exceeds the difference between r and ρ (for logarithmic utility), he will become a debtor, because his consumption will grow at a lower rate than income.[15] However, at the aggregate level, the growth rate in income must be equal to (r-ρ) as is perfectly clear from equation 30 in Appendix 2. The equilibrium rate of interest will adjust to ensure this condition.

As can be seen, the simple economy presented here is characterized by the fact that the growth rate in aggregate income is always lower than the real rate of interest provided that the subjective discount rate is positive. Thus, from the point of view of standard growth theories, this economy is always dynamically efficient, if people prefer the given satisfaction to be delivered as soon as possible (i.e. ρ>0). Recalling the implications from section 3.1.3, for constant money supply the nominal rate of interest must be always positive in this economy. This statement holds for any θ, not only θ=1 assumed here, since the expression (1+r)/(1+ρ) or (r-ρ) in equation (30) in Appendix 2 will be only modified by exponent (1/θ) or denominator θ, leaving the growth rate in output below r. (not true)

This is a direct outcome of a logarithmic utility and positive subjective discount rate. As we will see, this combination perfectly meets the requirement for a dynamically efficient economy.

---

[14] According to empirical studies (???), growth rate of consumption is tightly connected with the growth rate of income. Several theoretical models were developed to address this issue that is at variance with the standard neoclassical model presented here. See e.g. ???

[15] One should not be confused by the fact that a low growth rate of consumption leads to a borrowing position of the individual. We have to realize that this does not say anything about the absolute size of consumption in either period. If present consumption is close to future consumption and if the income stream is increasing, present consumption must exceed present income, which results in negative saving on the part of this individual.

In the following part, we will relax the assumption that the subjective discount rate is identical for all individuals in the economy. Suppose that there are two groups of people with different rates of time preference in sense two. Group A has the discount rate of ρ_A, group B of ρ_B. Suppose that the first group is more patient than the second group, hence $\rho_A<\rho_B$. Obviously, this assumption oversimplifies the real world, because not only the subjective discount rate differs among individual people, but it is changing on the individual basis as well. However, as we will see, even this simplification may provide us with key insights.

Panels a) and b) in Figure no. 31 display the equilibrium of an economy populated by two groups of people with different ρ, but (for simplicity) constant flow of income. If there were no intertemporal market, each group would consume its income in every period (point A_1 and A_2). The creation of the intertemporal market will make everybody better off, because the optimum of a representative individual of each group (E_1 and E_2) lies on a higher indifference curve. The interest rate must adjust such that the positive saving of group A (more patient people) is exactly the same as the negative saving of group B (less patient people), hence $Y - C_0^{A*} = C_0^{B*} - Y$. The equilibrium rate of interest will be between ρ_A and ρ_B and its precise value can be found in Appendix 2, in sections D and E. As is obvious from that mathematical treatment, the natural rate of interest does not depend on the level of income, if it is constant and the same for all individuals (section D). However, if the size of the constant income stream varies across individuals, the real rate of interest is affected in a sense that the size of income of the particular agent gives relative weight to the subjective discount rate of this agent in determining the size of the real interest rate. Nonetheless, the limits are determined by ρ_A and ρ_B and no level of income can push the interest rate outside these limits. And finally, it is obvious that the less patient agents will be characterized by a decreasing shape of their consumption flow, because their subjective discount rate is higher than the interest rate, whereas the more patient individuals will consume relatively more in the future, since their rate of time preference (in sense two) is lower than the rate of interest.

The foregoing approach might be generalized for n possible values of ρ. The natural rate of interest is then so adjusted that the aggregate level of saving is zero. At the same time, every individual in the economy is maximizing his or her lifetime utility at the point where the MRS is equal to $(1+r_E)$. Or alternatively, where the marginal rate of time preference, $\varepsilon_i \equiv (1+\rho_i)u'(C_{0i})/u'(C_{1i})-1$, is equal to the equilibrium natural rate of interest r_E. The optimum flow of consumption of each individual can be easily determined, as well as the fact whether the individual is a lender, borrower or does not enter the intertemporal market at all.

In the next discussion, we will relax the assumption that the time shape of the flow of income is the same for all people. Figure no. 32 illustrates a situation where the income stream of individual A is decreasing ($Y_0^A>Y_1^A$) and that of individual B is increasing ($Y_0^B<Y_1^B$). The real rate of interest must be so adjusted that the aggregate saving is zero. Compared with the previous example in Figure no. 31, the equilibrium real interest rate might be even negative, if the income stream of one group is decreasing sharply enough. The mathematical solution can be found in the benchmark example of Appendix 2, equations 1-16.

In our example in Figure no. 32, the patience of individual A (due to low ρ_A) is supported by a falling income over time, whereas high impatience of individual B (due to high ρ_B) is enhanced by a rising income stream. Thus, at the income endowment point of individual A (point A^A) the MRS is very low, whereas in case of individual B (point A^B) it is very high (see the dashed lines at A^A and A^B). We can say that the individual A has a very low time preference in sense one (it might be even negative, if MRS<1 at point A^A) and the individual B has a very high time preference in sense one at point A^B. However, as we already know, the time preference in sense one (i.e. the subjective exchange ratio between present goods and future goods represented by MRS) is an endogenous concept and it eventually depends on the optimum point that is posited at the highest possible indifference curve that touches the budget line. In market equilibrium, the natural rate of interest is so adjusted that positive saving of patient individuals A ($Y_0^A - C_0^{A*}$) is perfectly offset by negative saving of impatient individuals B ($C_0^{B*} - Y_0^B$). Moreover, at the optimum (i.e. lifetime utility maximizing) points of both groups and in the eventual market intertemporal equilibrium, the rates of time preference (in sense one), or the rates of impatience as I. Fisher would call it, must be the same for all individuals and they must be equal to the equilibrium real rate of interest (i.e. $MRS_A - 1 \equiv \varepsilon_A = r_E = \varepsilon_B \equiv MRS_B - 1$). Thus, the market process leads to the equalization of time preferences (in sense one) of various individuals regardless of their subjective discount rates (i.e. time preference in sense two) and the time shape and size of their income streams. The coordinating mechanism is due to the adjustment of the real rate of interest that ultimately guarantees that the objective exchange ratio between present goods and future goods is perfectly in accordance with the subjective exchange ratio of each individual.[16]

Both Mises (???) and Rothbard (???) wrote about the eventual equalization of the rates of time preference among various individuals. It is quite difficult to imagine a different interpretation than the adjustment of the individuals´ MRSs. However, since MRS can take on any value, greater weight might be put on future goods compared with present goods. Thus, the theory of Mises and Rothbard assuming a-priori positive time preference (in sense one), i.e. a-priori positive premium on the part of present goods, cannot be correct.

As a final note, let us discuss the optimum time shape of consumption of the individuals from the previous example. The answer is not so clear cut as before, as it depends on the eventual size of the real interest rate r. The problem is that compared with Figure no. 31 real interest rate is not bound by the interval determined by individual subjective discount rates (ρ_A, ρ_B), because a non-constant time shape of the income streams can move it to any level (see equation 16 in Appendix 2). Thus, r might be higher than ρ_A and ρ_B, it can be lower than both levels or it can be in between (as in Figure no. 32). As a result, the optimum flow of consumption of each individual can take any time shape. The wide variety of possible outcomes is presented in Appendix 2, sections 1-3.

As can be seen in the figures in Appendix 2, almost any combination is possible. The most noteworthy observations are as follows: An increasing time shape of income and higher subjective discount rate lead to a borrowing position. If they operate against each other, the eventual position depends on their relative strength.[17] Next, increasing income streams raise the equilibrium interest rate above both subjective discount rates, which results in the fact that the time shapes of consumption flows are also increasing (see section 6 in Appendix 2). Decreasing time shapes of income would imply a decline in the interest rate below both subjective discount rates. This would lead in turn to a decreasing time shape of consumption. The natural rate of interest might even fall below zero if the general decline in income is sharp enough. And finally, the natural rate of interest might be stabilised between the discount rates of various individuals, if the income streams are of the opposite time shapes or if they are constant over time. A perfectly smoothed consumption stream of an individual might be reached if the market interest rate is equal to his subjective discount rate, which seems to be

---

[16] Both Mises (???) and Rothbard (???) write about the eventual equalization of the rates of time preference among various individuals. It is quite difficult to imagine a different interpretation than the adjustment of the individuals´ MRSs. However, since MRS can take on any value, greater weight might be put on future goods compared with present goods. Thus, the theory of Mises and Rothbard assuming a-priori positive time preference (in sense one), i.e. a-priori positive premium on the part of present goods, cannot be correct.

[17] In section 4 of Appendix 2, an individual with a lower subjective discount rate (i.e. individual A) is a borrower due to his sharply increasing flow of income. Thus, relatively low time preference (in sense two) does not guarantee a net lending position, if the flow of income of the individual is growing at a sufficient rate (or if it is falling at a lower rate) compared with others.

an exception rather than a rule. Consumption will not be smoothed even if both income streams are perfectly smoothed provided that the subjective discount rates differ (see Figure no. 31 above and Appendix 2, section 5). In such a case, the consumption stream of a more patient individual will be increasing, whereas that of the less patient individual will be decreasing, since the equilibrium interest rate will be stabilized in the interval between ρ_A and ρ_B.

However, two combinations of consumption streams of A and B are impossible. Because $\rho_A < \rho_B$, if the consumption stream of A is decreasing ($r < \rho_A$), that of B cannot be increasing (since $r>\rho_B$ is inconsistent with $\rho_A < \rho_B$ and $r < \rho_A$). And conversely, if the consumption stream of B is increasing ($r>\rho_B$), that of A cannot be decreasing (since $r<\rho_A$ is inconsistent with $\rho_A < \rho_B$ and $r > \rho_B$).

It should be stressed that the most common situation is probably an increasing time shape of income in general, positive real rate of interest that exceeds both the average growth rate in income and ρ_A and ρ_B, and a borrowing position of individuals with higher ρ (i.e. B with ρ_B) and a lending position of individuals with lower ρ (i.e. A with ρ_A).[18] This situation is portrayed in Appendix 2, section 6.

As can be seen, the heterogeneity of agents (both as regards income streams and subjective discount rates) leads to the creation of the intertemporal market, where borrowers and lenders exchange present goods for future goods. As is shown in Appendix 2, section 7, the intertemporal market will exist, even if the natural rate of interest is zero (or negative, see equation 17 in Appendix 2). Thus, we have constructed another theoretical model that is at odds with the Garrison´s critique of the neoclassical theory. In our model, all individuals prefer the given want to be satisfied as soon as possible (both ρ_A and ρ_B are greater than zero) so the key Misesian maxim is not violated. However, the natural rate of interest is zero, i.e. present goods are not preferred to future goods. Yet, a vivid intertemporal market has been created (i.e. some people lend and others borrow) even without the existence of interest. Thus, the critique of R. Garrison might be easily overcome.

In actual world, people differ in subjective discount rates, utility functions and shapes of their income streams. Here, we separated each factor of the natural rate of interest in order to analyze its specific impact. However, the main message of our analysis is clear; the natural rate of interest is a complicated function of the discount rates of individuals (ρ_i), shapes of their income streams (Y_{0i},Y_{1i}) and the shape of their utility functions ($\theta_i,\ldots$). It may take on any value. The most plausible is its positive value, because people prefer present satisfaction to future satisfaction ($\rho_i>0$). However, if the (expected) income stream of a considerable part of population is decreasing, it may fall below zero. The statement of the Misesian PTPC, that the natural rate of interest is solely determined by the pure time preference, holds only under very special circumstances.

Further extensions of the analysis of the natural rate of interest are also possible. We can combine the previous two sections – man can face investment opportunities and he may also have an exogenous income endowment in both periods. Furthermore, if the leisure time enters the utility function, the income stream is no longer exogenous and the key parameters of the utility function would determine not only the shape and position of the indifference curves but also the position of the income endowment. And finally, the assumption of a single-good

[18] One may wonder why the real interest rate in normal conditions is not between ρ_A and ρ_B. More on this will be said in the final section. The fundamental reason is, however, an increasing time shape of the (aggregate) income stream (i.e. the first Bohm-Bawerkian cause for interest) that drives up the interest rate above the subjective discount rate of even the most impatient individual (in our case B with ρ_B).

economy can be relaxed and an effort (maybe futile) to find the natural rate of interest for an n-good economy might be carried out.

This section will be concluded with the first extension. Utility function that includes leisure (or labour effort) will be postponed to Appendix 3. The third one, an n-good economy, was only briefly mentioned in the previous section, since examining the natural rate of interest in such an economy would require much deeper investigation and more time, space and even intellectual abilities than the author of the present study seem to be endowed with.

Figure no. 33 portrays an economy with investment opportunities and the income endowment also in the second period. This may represent an economy with pure labour (and land) available in both periods, in which the labour in the first period can be used in a longer production process, whereas labour in the second period might be employed only in the short process.[19] The natural rate of interest in this economy is co-determined by subjective factors (ρ,θ) and the productivity of capital (point E), or by the income stream and subjective factors for a different set of parameters (point A, not shown as the optimum in our picture). In the later case, the entire supply of present labour would be used only in short production processes, whereas in the former case present labour is employed also in longer processes, which reduces present output on behalf of future output. Figure no. 33 represents an economy with identical individuals. However, if agents differ in ρ (and θ) and in their investment opportunities, the individual investment need not equal individual saving. As a result, man can borrow from more patient agents to make even higher investment than in Figure no. 33 and fill the lack of saving by a loan from the others. This situation might be represented by Figure no. 28, if the extreme income endowment A is moved along the budget line from the horizontal axis closer to the vertical axis. However, the beginning of the investment opportunity line will stay at the horizontal axis (Stigler ???:???). The level and amount of optimum consumption, investment, saving and loan can be easily described by a similar system of points as in Figure no. 28.

It can be also assumed that the income endowment is easily storable (Figure no. 34, panel a) or it might have constant productive power as in panel b. The natural rate of interest then depends on the specific shape of the income stream. In Figure 34a, the natural rate of interest r_E is determined by the subjective discount rate ρ and the shape of the income stream (i.e. by the time preference in sense one – MRS), not by productivity. If r was lower than r_E (e.g. $r = 0$ %), the excess of borrowing would immediately emerge. This will in turn drive up the interest rate back to r_E. Similar analysis holds for panel b. However, in panel b) there is a higher chance that the natural rate of interest will be determined by constant productivity, because the insignificance of productivity requires much sharper increase in income endowment over time (i.e. $Y_1 >> Y_0$). It means that the amount of present original factors of production that might be used in a longer process (that exhibits constant positive productivity) must be quite small.

Yet, in Figure no. 35 the natural rate of interest is definitely zero (i.e. determined by productivity) due to the fact that the income stream is strongly decreasing over time and the good in question is storable. Again, even though people exhibit positive time preference in sense two ($\rho>0$), the natural rate of interest (and time preference in sense one) is zero ($MRS-1 \equiv \varepsilon = r = 0$). Saving takes place in this economy ($Y_0 - C_0^*$), which will not be, however, traded in the intertemporal market. It will take the form of a simple storage of non-perishable goods held to the poorly endowed future. Notice that the negative (real) natural interest rate could only emerge if the representative good was perishable (the slope of the linear line was below 1) and also in very low supply in the future ($Y_1 << Y_0$).

[19] This model is outlined in section C of Appendix 3.

Figures

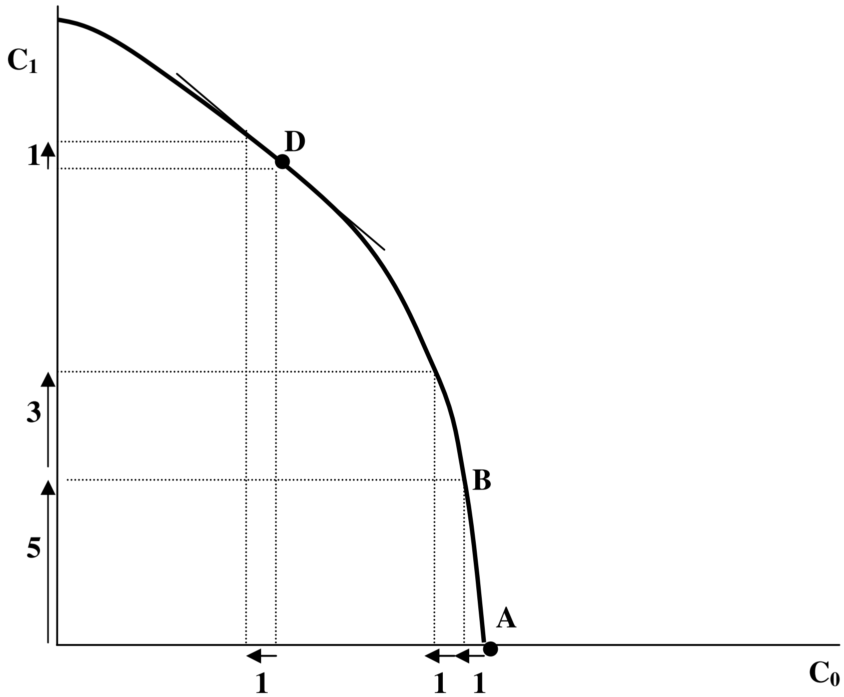


Figure no. 20, Greater fruitfulness of longer methods and their diminishing marginal productivity

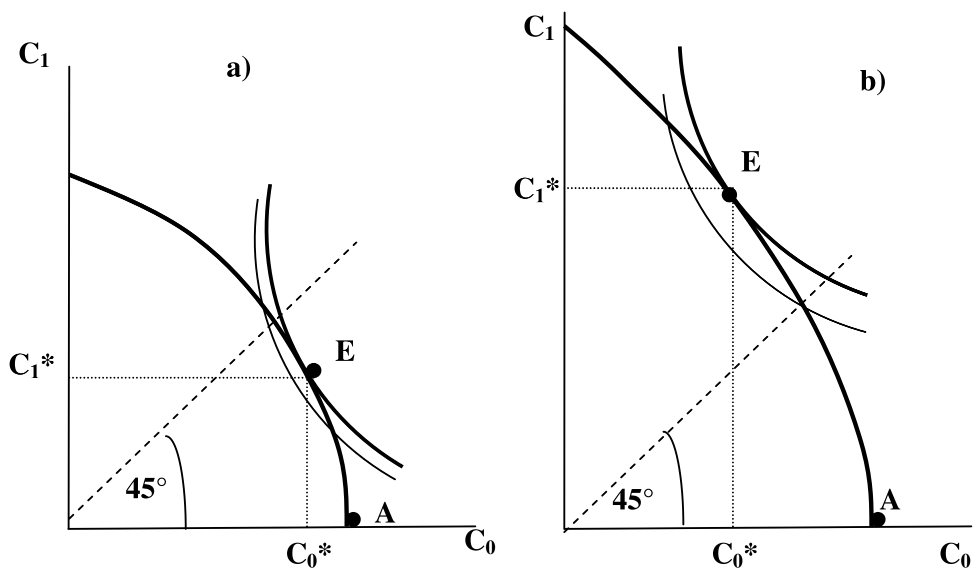


Figure no. 21 Equilibria for various time preferences and marginal productivity schedules

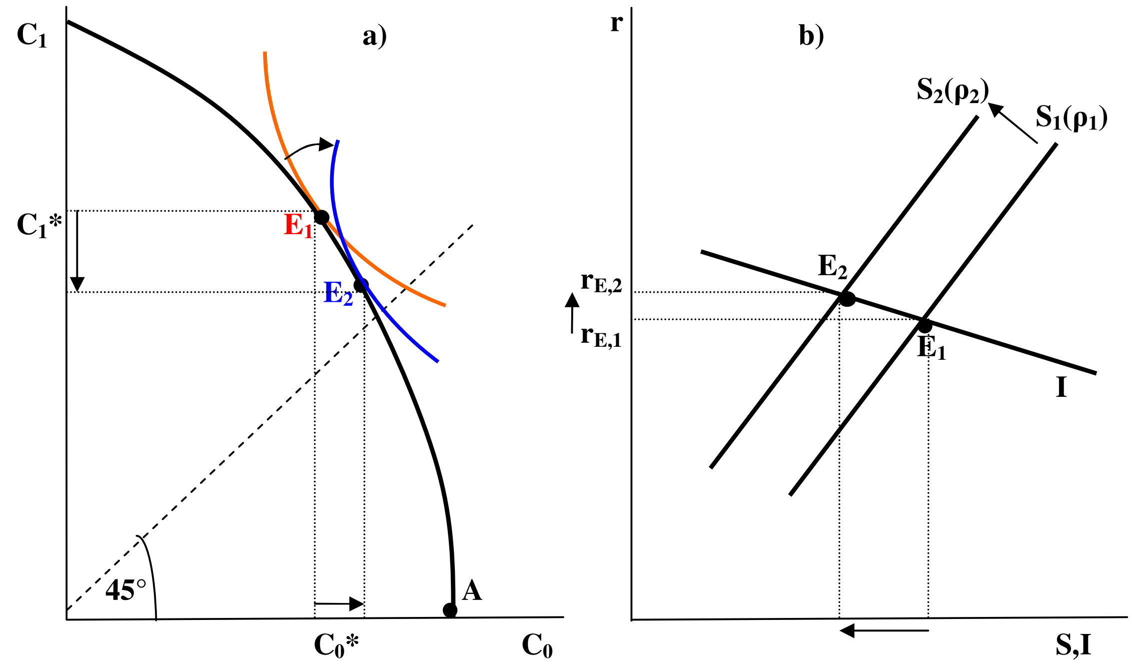


Figure no. 22 Increase in time preference (in sense two) and the impact on the natural rate of interest, if the marginal productivity scheme diminishes slowly

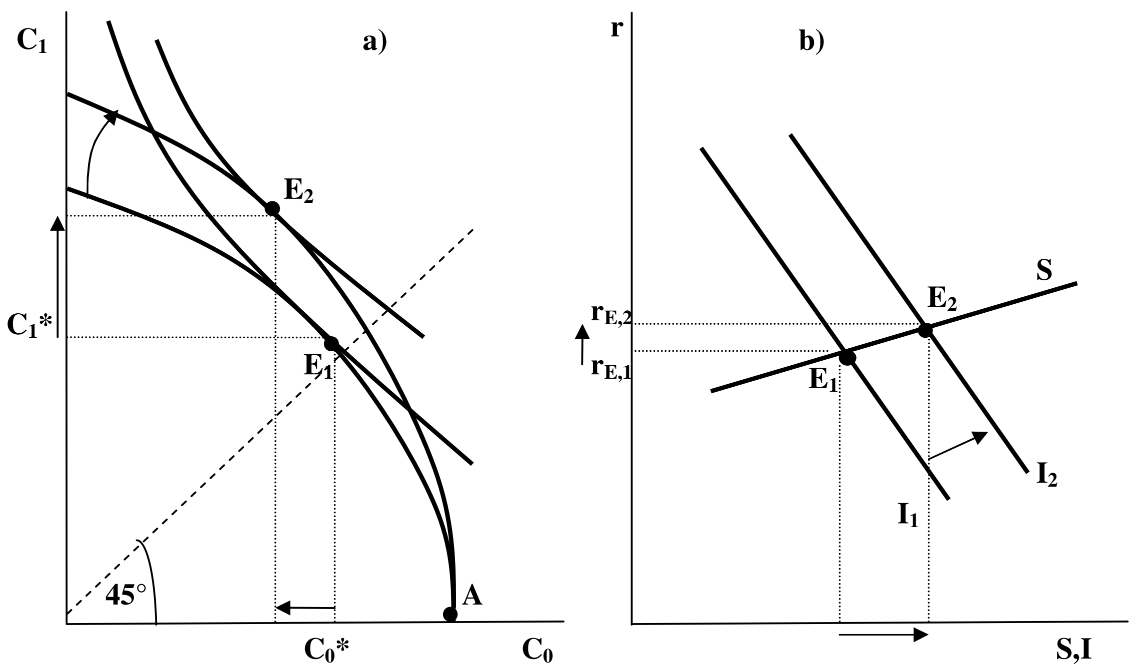


Figure no. 23 Increase in productivity and the impact on the natural rate of interest, if the elasticity of substitution is very high (low θ)

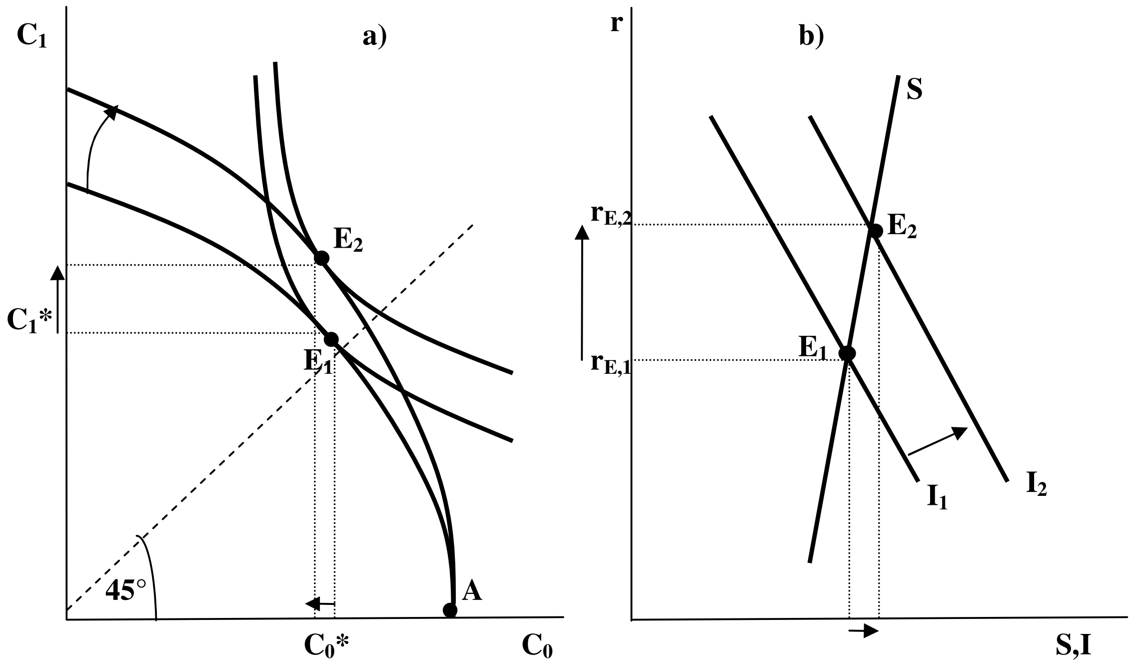


Figure no. 24 Increase in productivity and the impact on the natural rate of interest, if the elasticity of substitution is low (higher θ)

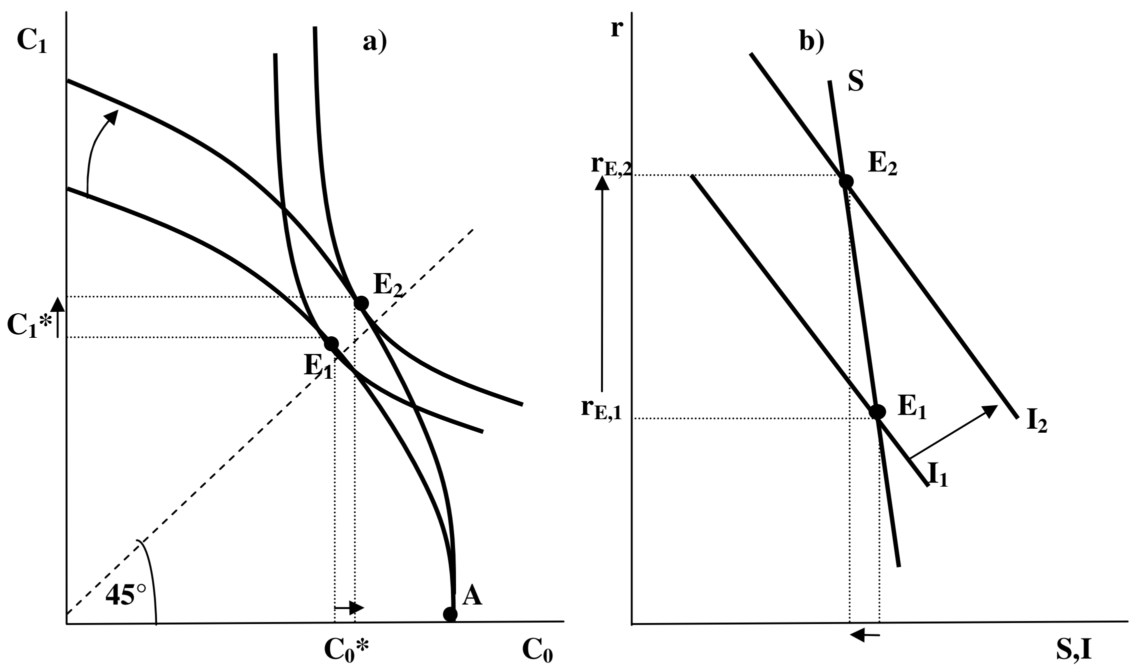


Figure no. 25 Increase in productivity and the impact on the natural rate of interest, if the saving curve is decreasing (θ>1)

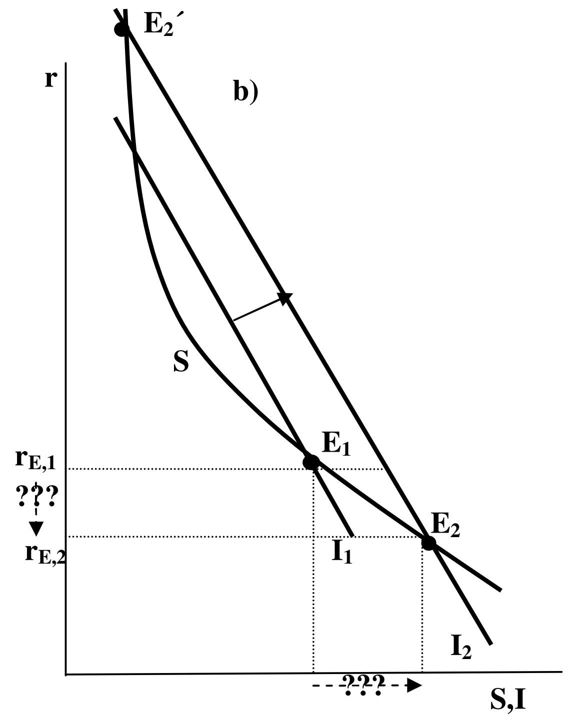


Figure no. 26B Increase in productivity and the impact on the natural rate of interest, if the saving curve is decreasing and more elastic (θ>>1) than the investment curve. Multiple equilibria.

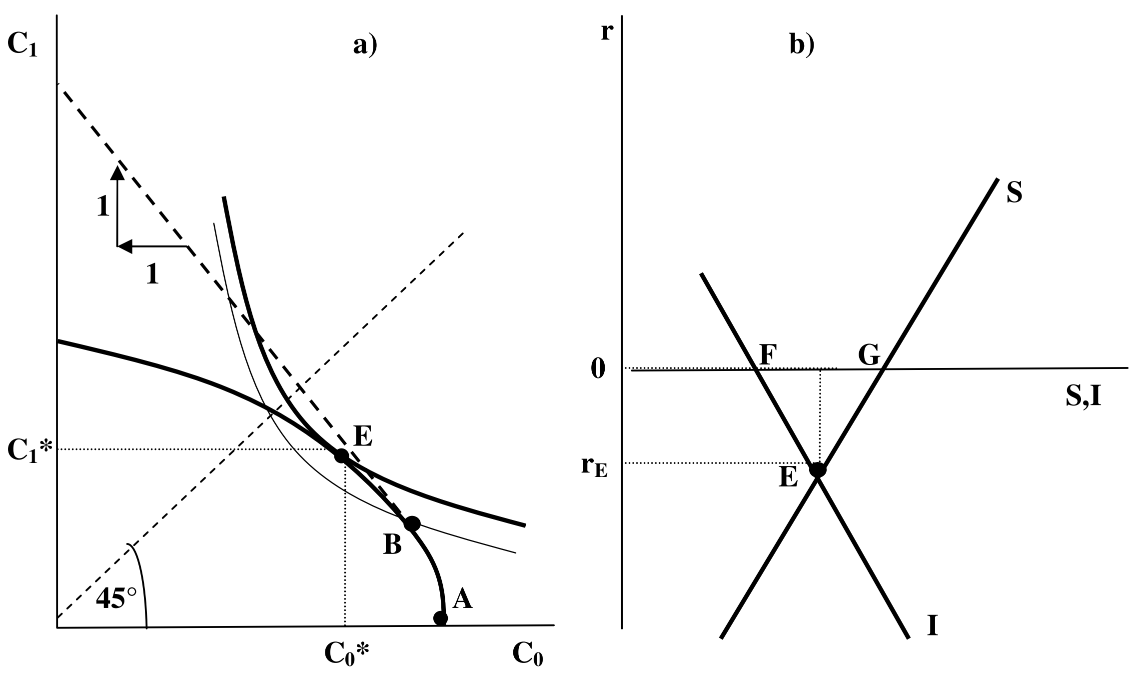


Figure no. 27 Negative natural rate of interest

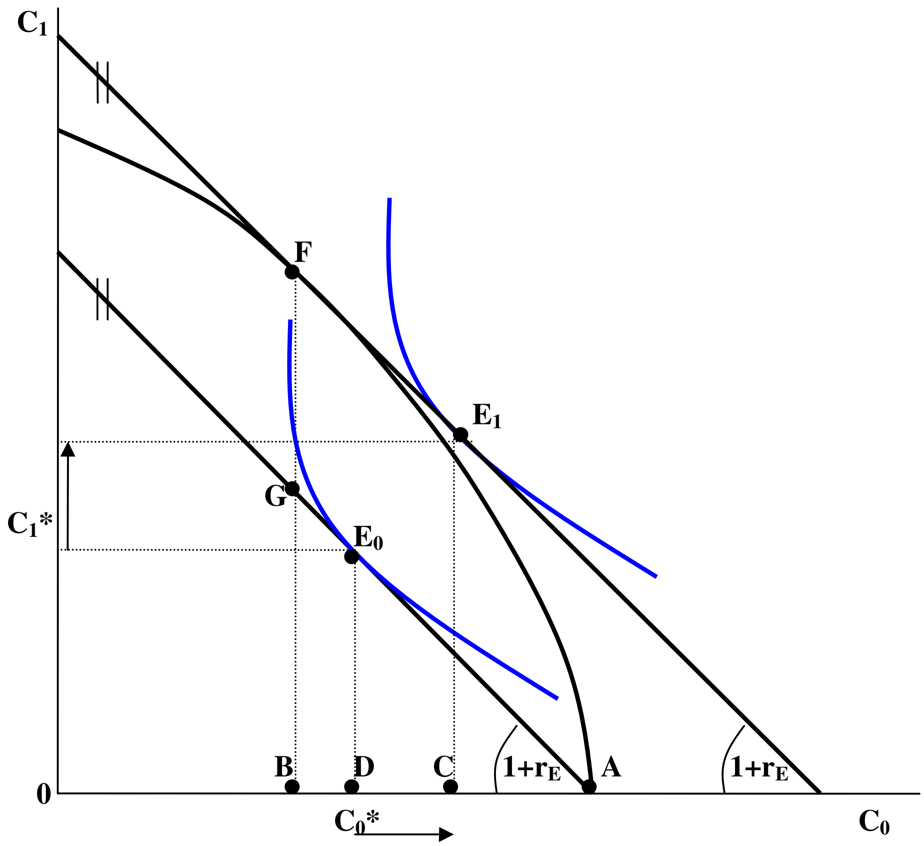


Figure no. 28 An individual in an economy with heterogeneous agents

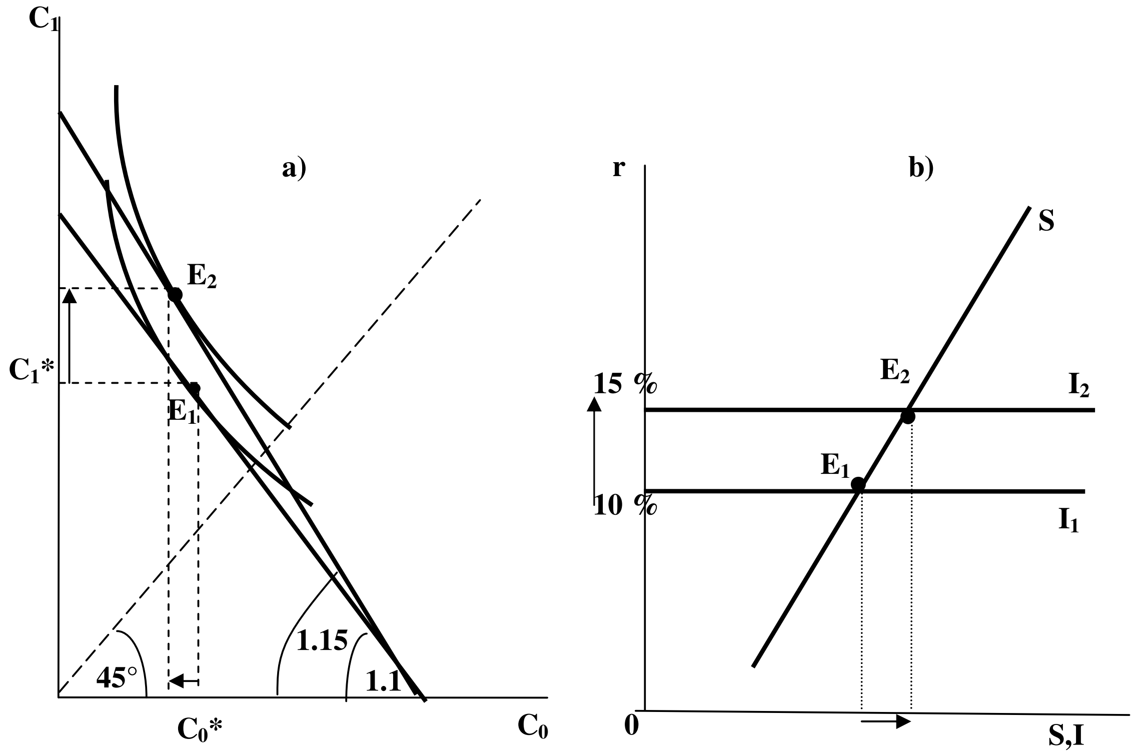


Figure no. Rothbard 1 Increase in productivity in the economy with constant MPK

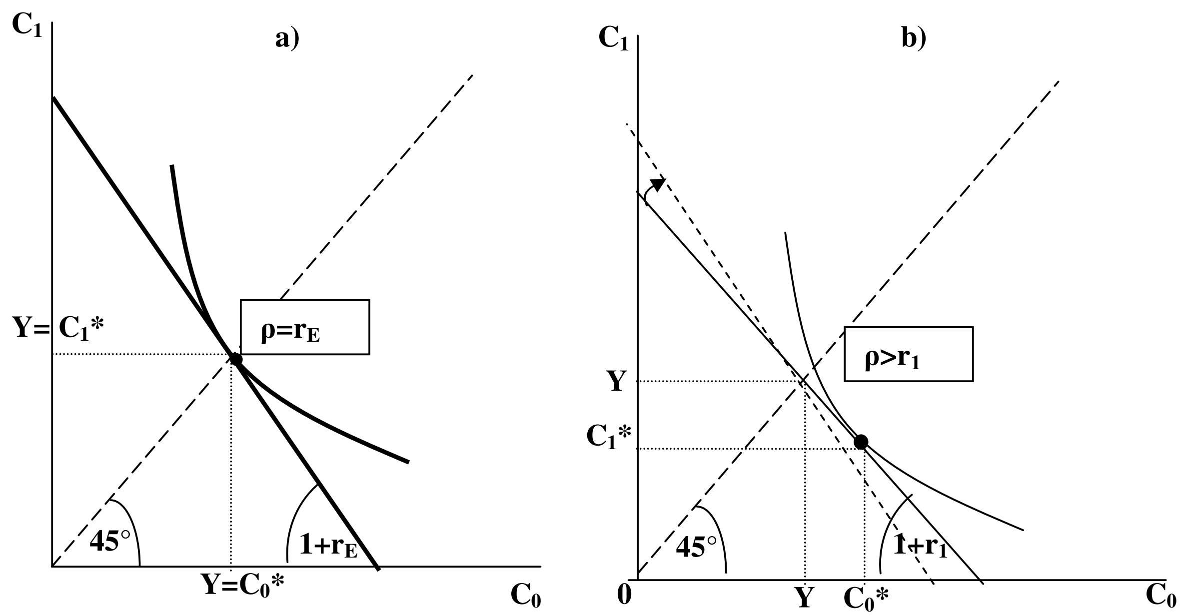


Figure no. 29 Natural rate of interest and a constant flow of income

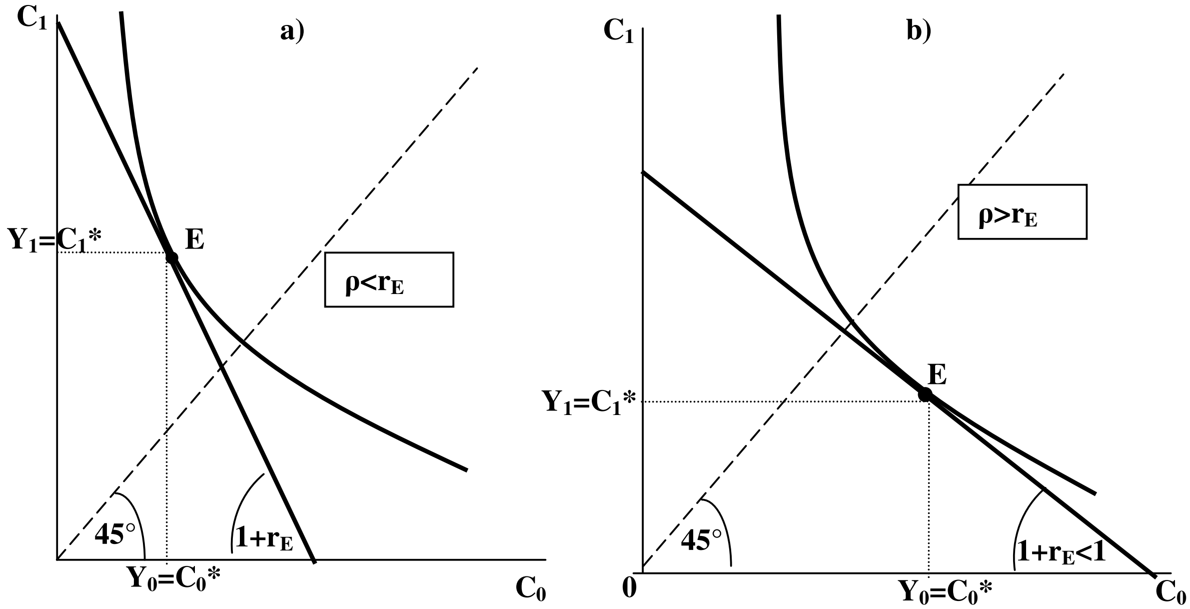


Figure no. 30 Increasing (a) and decreasing (b) income stream and the corresponding natural rate of interest

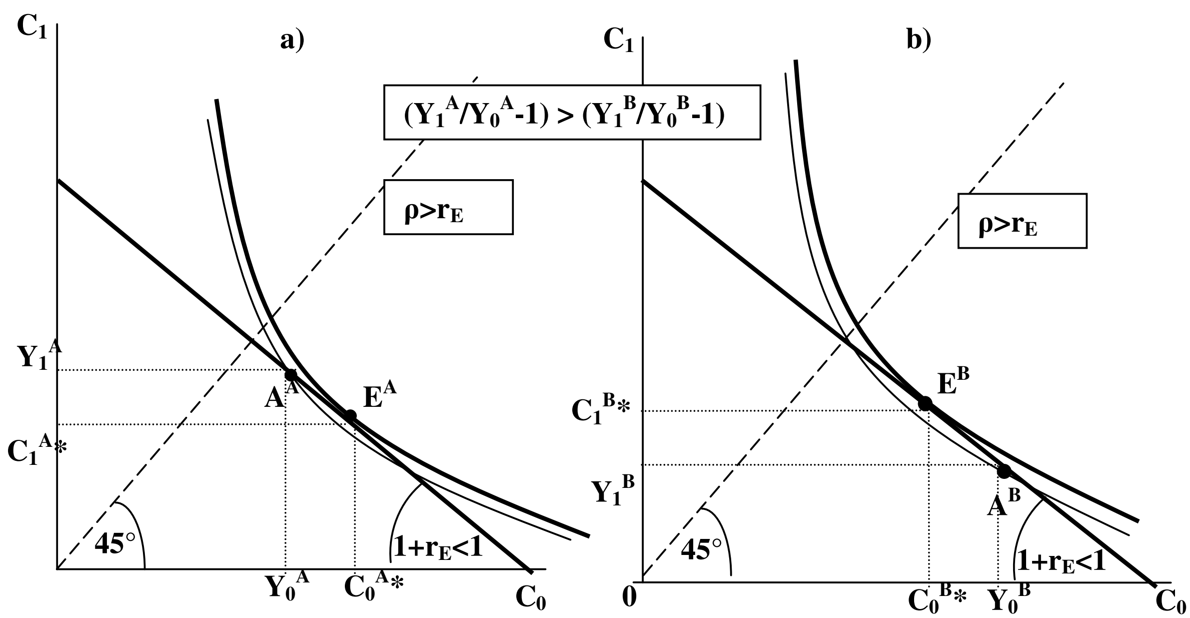


Figure no. 30B Creditors might exist even if the natural rate of interest is negative (panel b)

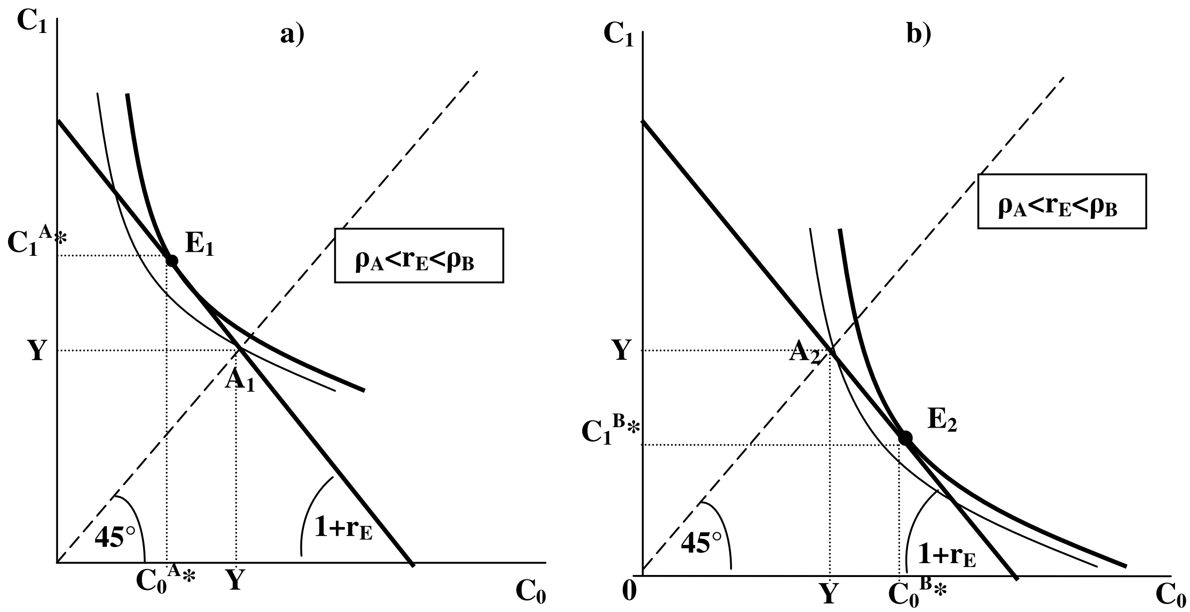


Figure no. 31 More patient (a) and less patient (b) agents and the corresponding natural rate of interest.

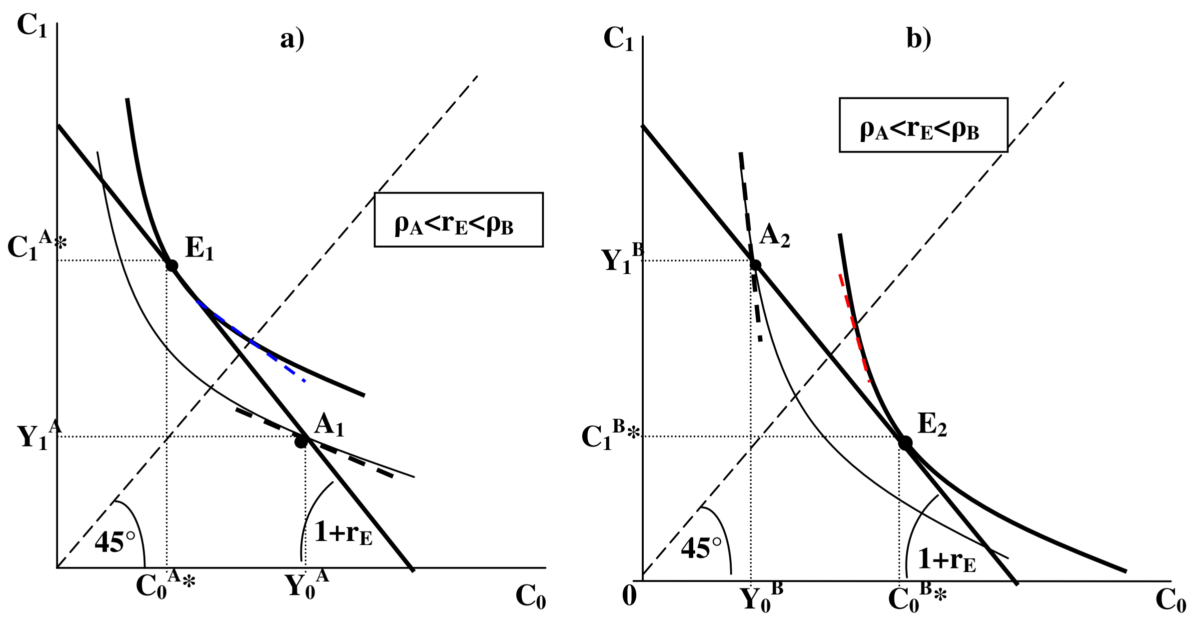


Figure no. 32 Equalisation of the rates of time preference (in sense one) with the real interest rate and also among individuals, regardless of the time shape of their income stream and the size of their subjective discount rate. (!!! Corner solution Seager !!!)

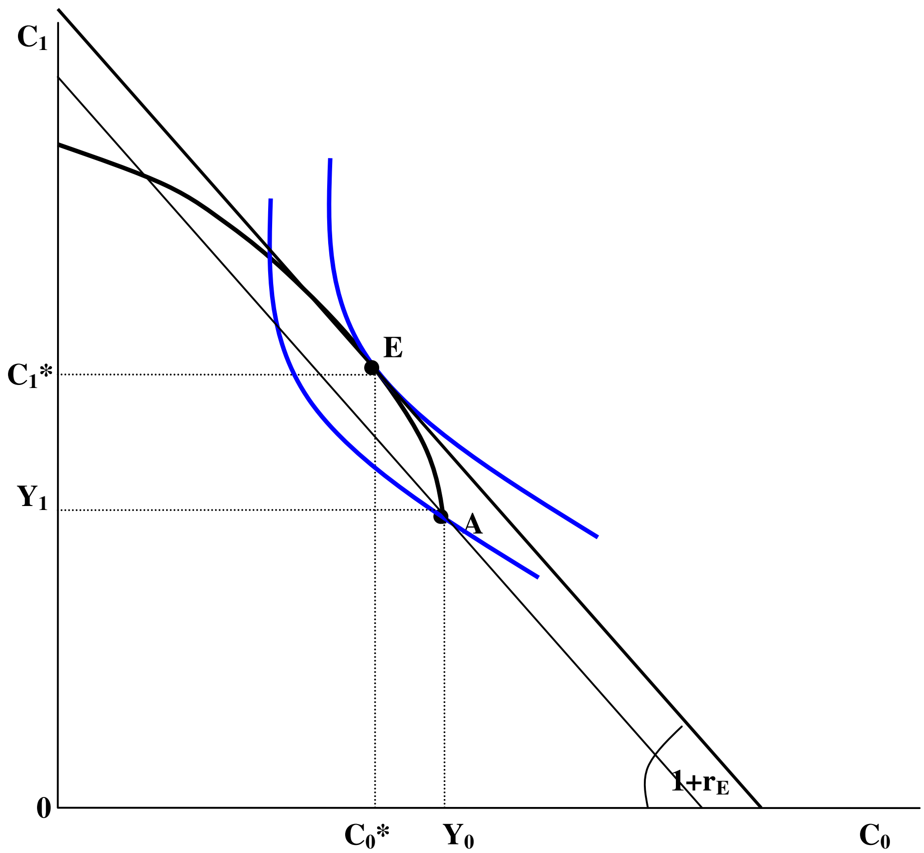


Figure no. 33 Natural rate of interest in the economy with income/labour(and land) endowment and investment opportunities.

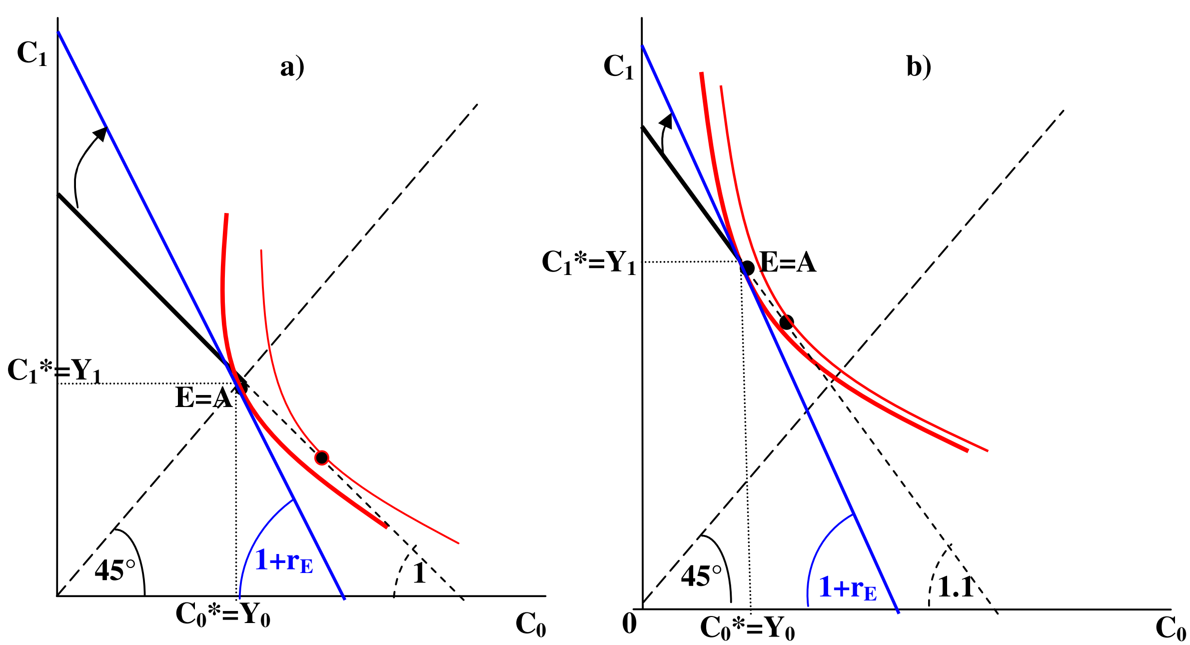


Figure no. 34 Storable income endowment that is positive in both periods. Natural rate of interest determined by the time preference in sense one (i.e. MRS)

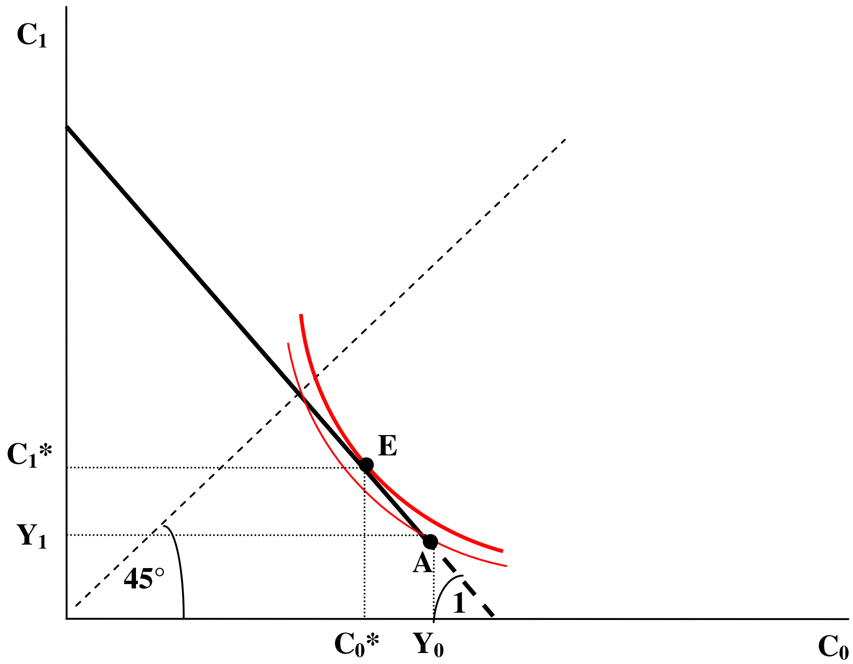


Figure no. 35 Storable income endowment that is positive in both periods. Case of zero interest

Appendix 2:

Consider a representative consumer A maximizing his life-time utility in a simple two-period model (Equation 1). For simplicity, assume that $\theta=1$, hence the utility function is logarithmic. Equation (2) represents his intertemporal budget constraint. Y_0 and Y_1 stand for his (labour) income in the present and in the future.

$$U = \ln C_0^A + \frac{1}{1+\rho_A} \ln C_1^A \qquad (1)$$

$$C_0^A + \frac{1}{1+r} C_1^A = Y_0^A + \frac{1}{1+r} Y_1^A \qquad (2)$$

Set up a simple Lagrangian function and solve for the first order conditions (FOC).

$$L = \ln C_0^A + \frac{1}{1+\rho_A} \ln C_1^A + \lambda (Y_0^A + \frac{1}{1+r} Y_1^A - C_0^A - \frac{1}{1+r} C_1^A) \qquad (3)$$

FOC:

$$\frac{\partial L}{\partial C_0^A} = \frac{1}{C_0^A} - \lambda = 0 \qquad (4)$$

$$\frac{\partial L}{\partial C_1^A} = \frac{1}{1+\rho_A} \frac{1}{C_1^A} - \lambda \frac{1}{1+r} = 0 \qquad (5)$$

$$\frac{C_1^A}{C_0^A} = \frac{1+r}{1+\rho_A} \qquad (6)$$

Equation (6) represents the Euler equation for this problem. It describes the optimal allocation of consumption over time. By substituting it into the IBC (eq. 2) and after simple manipulations, we get an optimum consumption in the present and in the future (eq. 7 and 8):

$$C_0^A * = \frac{(1+r)Y_0^A + Y_1^A}{1+r} \frac{1+\rho_A}{2+\rho_A} \qquad (7)$$

$$C_1^A * = \frac{(1+r)Y_0^A + Y_1^A}{2+\rho_A} \qquad (8)$$

Suppose for simplicity that there are only two individuals in the economy (or two groups of representative individuals). They differ in their income streams and their subjective discount rates. We could of course extend the analysis by including n individuals. However, this will only complicate things without giving more insight that might be obtained even with a simple example with two individuals. Thus, the optimum of individual B is described by similar equations as in (7) and (8).

Equations (9) and (10) characterize resource constraints in the economy in the present and in the future. Equation (9) basically states that the aggregate consumption at time 0 may not exceed the aggregate income at time 0. An alternative interpretation is that saving/borrowing of A must be equal to borrowing/saving of B. In other words, in the endowment economy without investment opportunities, aggregate saving must be equal to zero. Equation (10) is a corresponding aggregate constraint in the future. Both constraints might be easily constructed for n individuals, yet we will adhere to a simple 2-person model.

$$C_0^A * + C_0^B * = Y_0^A + Y_0^B \tag{9}$$

$$C_1^A * + C_1^B * = Y_1^A + Y_1^B \tag{10}$$

Our system consists of 5 unknowns (C_0^{A*}, C_1^{A*}, C_0^{B*}, C_1^{B*}, r) and 6 equations (7 and 8 both for A and B, and 9 and 10). Thus, one equation is not independent. Let us use (10) and substitute optimum consumption levels from equation (8) for both individuals. This yields:

$$\frac{Y_0^A + rY_0^A + Y_1^A}{2+\rho_A} + \frac{Y_0^B + rY_0^B + Y_1^B}{2+\rho_B} = Y_1^A + Y_1^B \tag{11}$$

$$(Y_0^A + rY_0^A + Y_1^A)(2+\rho_B) + (Y_0^B + rY_0^B + Y_1^B)(2+\rho_A) = (Y_1^A + Y_1^B)(2+\rho_A)(2+\rho_B) \tag{12}$$

$$r\left[Y_0^A(2+\rho_B) + Y_0^B(2+\rho_A)\right] = (Y_1^A + Y_1^B)(2+\rho_A)(2+\rho_B) - (Y_0^A + Y_1^A)(2+\rho_B) - (Y_0^B + Y_1^B)(2+\rho_A)$$
(13)

$$r = \frac{(Y_1^A + Y_1^B)(2+\rho_A)(2+\rho_B) - Y_1^A(2+\rho_B) - Y_1^B(2+\rho_A) - Y_0^A(2+\rho_B) - Y_0^B(2+\rho_A)}{Y_0^A(2+\rho_B) + Y_0^B(2+\rho_A)} \tag{14}$$

$$r = \frac{(Y_1^A + Y_1^B)(2+\rho_A)(2+\rho_B) - Y_1^A(2+\rho_B) - Y_1^B(2+\rho_A)}{Y_0^A(2+\rho_B) + Y_0^B(2+\rho_A)} - 1 \tag{15}$$

$$r = \frac{Y_1^A(2+2\rho_A+\rho_B+\rho_A\rho_B) + Y_1^B(2+\rho_A+2\rho_B+\rho_A\rho_B)}{Y_0^A(2+\rho_B) + Y_0^B(2+\rho_A)} - 1 \tag{16}$$

As can be seen, equilibrium real interest rate r rises with higher future income (Y_1^A or Y_1^B), lower present income (Y_0^A or Y_0^B) and higher subjective discount rates (ρ_A or ρ_B). If we substitute r into (7) and compare C_0* with Y_0, we can decide whether the given individual is a lender or a borrower.

Furthermore, natural real interest rate may fall below zero, if the future income of individuals is relatively low compared with the present income. Hence, r<0 if:

$$Y_1^A(2+2\rho_A+\rho_B+\rho_A\rho_B)+Y_1^B(2+\rho_A+2\rho_B+\rho_A\rho_B)<Y_0^A(2+\rho_B)+Y_0^B(2+\rho_A) \quad (17)$$

A) For a constant flow of income and the same income for all individuals ($Y_0^A=Y_1^A=Y_0^B=Y_1^B=Y$) and for the same subjective discount rate ($\rho_A=\rho_B=\rho$), (16) will turn into:

$$r=\frac{Y(2+2\rho+\rho+\rho^2)+Y(2+2\rho+\rho+\rho^2)}{Y(2+\rho)+Y(2+\rho)}-1 \qquad (18)$$

$$r=\frac{2(2+3\rho+\rho^2)}{2(2+\rho)}-1 \qquad (19)$$

$$r=\frac{2+3\rho+\rho^2-2-\rho}{2+\rho} \qquad (20)$$

$$r=\frac{\rho(2+\rho)}{2+\rho} \qquad (21)$$

$$r=\rho \qquad (22)$$

Thus, for constant and identical income for all individuals and identical subjective discount rate (i.e. for a homogenous agent model in stationary conditions) the equilibrium real rate of interest is solely determined by the rate of time preference (in sense two) and it cannot fall below zero, unless ρ is negative. The level of income plays no role, if ρ itself is taken as an exogenous constant that does not depend on the size of Y.

Figure no. 1_A2 demonstrates how the new equilibrium r is established after an increase in the subjective discount rate. Higher ρ makes the indifference curve steeper at the 45º line (panel a). The new optimum lies to the right of the original one. This, however, creates an excess of demand for present goods over their supply (C_0^* - Y). Thus, the interest rate must go up to equalize the demand and supply of present goods. The new equilibrium is depicted in panel b. The new budget line is steeper, reflecting higher interest rate.

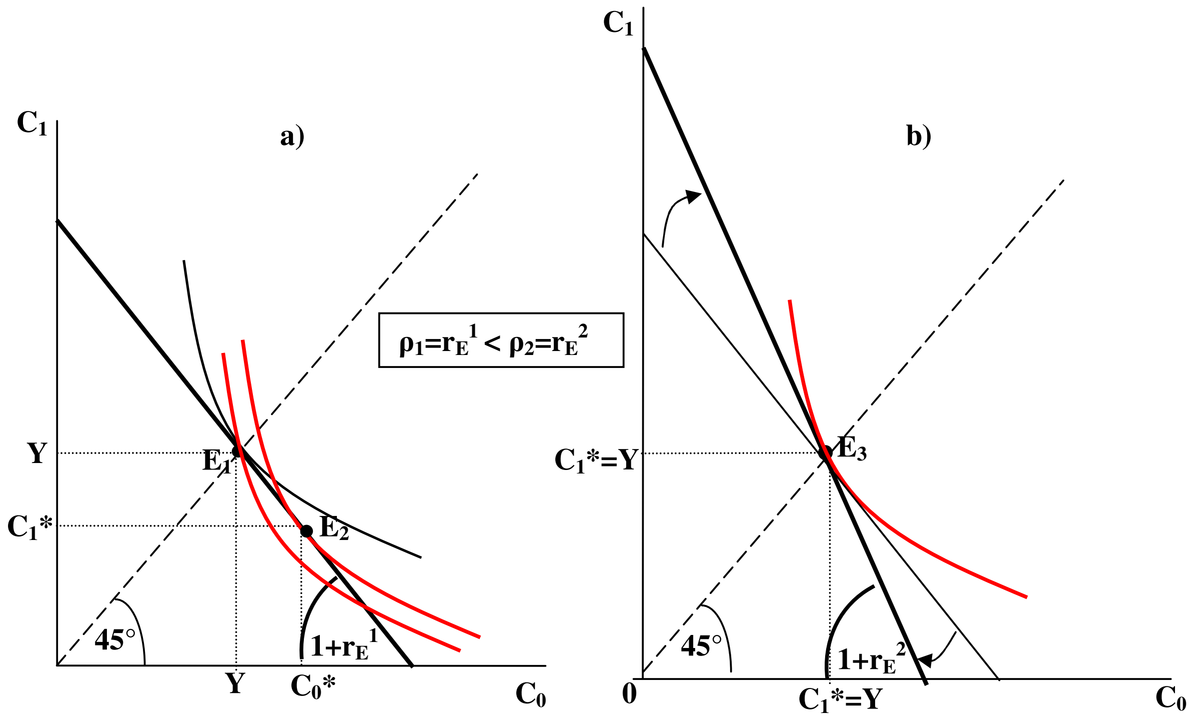


Figure no. 1_A1 Increase in the subjective discount rate will lead to higher interest rate.

B) If the subjective discount rate is the same for all individuals ($\rho_A=\rho_B=\rho$), if all have a constant flow of income but of different size (i.e. $Y_0^A = Y_1^A = Y^A$ and $Y_0^B = Y_1^B = Y^B$), (16) might be written as:

$$r = \frac{Y^A(2+2\rho+\rho+\rho^2)+Y^B(2+2\rho+\rho+\rho^2)}{Y^A(2+\rho)+Y^B(2+\rho)} - 1 \tag{23}$$

$$r = \frac{(Y^A+Y^B)(2+3\rho+\rho^2)-(Y^A+Y^B)(2+\rho)}{(Y^A+Y^B)(2+\rho)} \tag{24}$$

$$r = \frac{\rho(2+\rho)}{(2+\rho)} \tag{25}$$

$$r = \rho \tag{26}$$

As we can see, even if people have different size of income, its constancy over time leads to the fact that the equilibrium rate of interest depends only on the subjective discount rate.

C) If the subjective discount rate is the same for all individuals ($\rho_A=\rho_B=\rho$), but their flows of income differ being of any shape, (16) is modified to:

$$r = \frac{Y_1^A(2+2\rho+\rho+\rho^2)+Y_1^B(2+2\rho+\rho+\rho^2)}{Y_0^A(2+\rho)+Y_0^B(2+\rho)} - 1 \tag{27}$$

$$r = \frac{(Y_1^A + Y_1^B)(2+\rho)(1+\rho)}{(Y_0^A + Y_0^B)(2+\rho)} - 1 \qquad (28)$$

$$r = \frac{(Y_1^A + Y_1^B)(1+\rho)}{(Y_0^A + Y_0^B)} - 1 \qquad (29)$$

By substituting r to (7) we can determine, whether the particular individual is a debtor or a creditor. Debtors (e.g. individuals A) are characterized by the condition that the growth rate of their income stream (Y_1^A/Y_0^A-1) is higher than the growth rate of the income stream of creditors (Y_1^B/Y_0^B-1).

Because ($Y_0^A + Y_0^B$) and ($Y_1^A + Y_1^B$) are equal to the aggregate income in the economy in the given period, i.e. Y_0 and Y_1 respectively, (29) might be written as:

$$r = \frac{Y_1(1+\rho)}{Y_0} - 1 \qquad (30)$$

As in (16), the equilibrium real rate of interest is positively related to future income and the subjective discount rate and negatively related to present income. Furthermore, negative real rate of interest is possible (see 30 and 31), if the ratio of present income to future income is greater than (1+ρ) or alternatively, if the ratio of future income to present income is lower than the subjective discount factor $\beta \equiv 1/(1+\rho)$. In other words, future income must be sufficiently lower compared with the present income to achieve a premium of future goods over present goods.

$$\frac{Y_0}{Y_1} > (1+\rho) \qquad (31)$$

D) If people differ in their subjective discount rates, but have the same and constant flow of income, (16) might be written as:

$$r = \frac{Y(2+2\rho_A+\rho_B+\rho_A\rho_B) + Y(2+\rho_A+2\rho_B+\rho_A\rho_B)}{Y(2+\rho_B) + Y(2+\rho_A)} - 1 \qquad (32)$$

$$r = \frac{4+3\rho_A+3\rho_B+2\rho_A\rho_B}{4+\rho_A+\rho_B} - 1 \qquad (33)$$

$$r = \frac{2\rho_A+2\rho_B+2\rho_A\rho_B}{4+\rho_A+\rho_B} \qquad (34)$$

As in A), the equilibrium natural real rate of interest does not depend on income, if it is constant and the same for all individuals. Only the subjective discount rates matter. They raise the equilibrium rate of interest, which cannot fall below zero, unless they become negative too.

E) Consider heterogeneous agents with different subjective discount rates and different incomes that is, however, constant over time. Hence $Y_0^A = Y_1^A = Y^A$ and $Y_0^B = Y_1^B = Y^B$. (16) will take the form as follows:

$$r = \frac{Y^A(2+2\rho_A+\rho_B+\rho_A\rho_B)+Y^B(2+\rho_A+2\rho_B+\rho_A\rho_B)}{Y^A(2+\rho_B)+Y^B(2+\rho_A)} - 1 \tag{35}$$

$$r = \frac{Y^A(2+2\rho_A+\rho_B+\rho_A\rho_B)+Y^B(2+\rho_A+2\rho_B+\rho_A\rho_B)-Y^A(2+\rho_B)-Y^B(2+\rho_A)}{Y^A(2+\rho_B)+Y^B(2+\rho_A)} \tag{36}$$

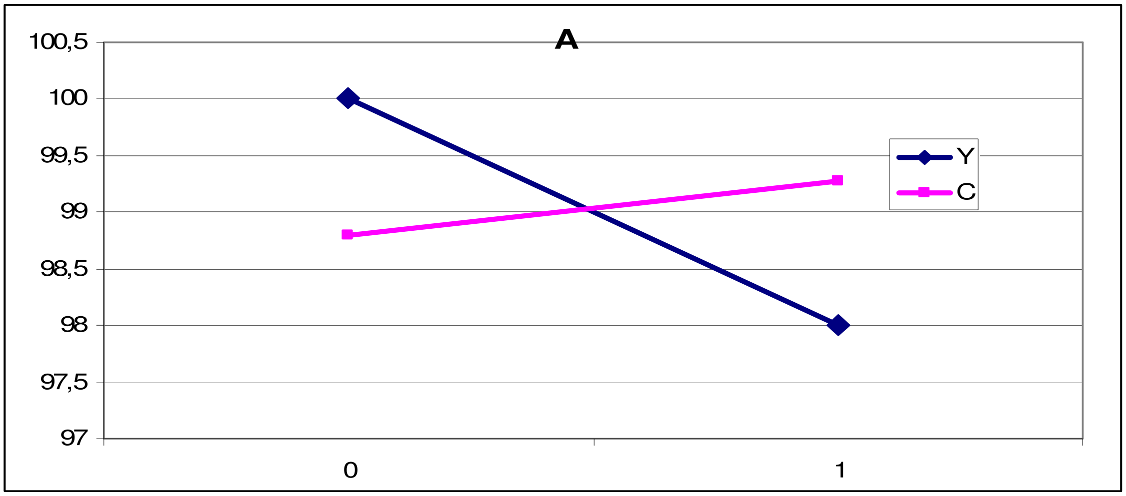
$$r = \frac{2\rho_A Y^A + 2\rho_B Y^B + \rho_A\rho_B Y^A + \rho_A\rho_B Y^B}{Y^A(2+\rho_B)+Y^B(2+\rho_A)} \tag{37}$$

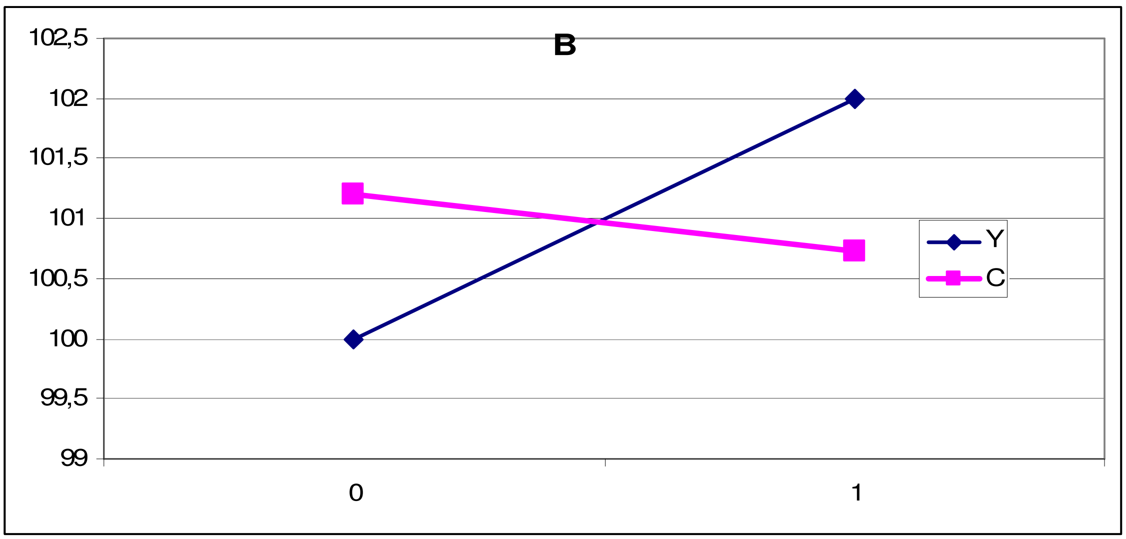
$$r = \frac{\rho_A Y^A(2+\rho_B)+\rho_B Y^B(2+\rho_A)}{Y^A(2+\rho_B)+Y^B(2+\rho_A)} \tag{38}$$

Again, the equilibrium real rate of interest cannot fall below zero, if the subjective discount rates are positive. Its value is between ρ_A and ρ_B. Moreover, the higher is the income of individual A compared with B, the closer is the real rate of interest to the subjective discount rate of individual A. Thus, in this case the size of the constant flow of income might affect the equilibrium real rate of interest, whose limits are, however, determined by particular subjective discount rates. Constant income flows of different size therefore give different weights to the particular discount rate in its determination of the equilibrium real rate of interest.

Simulations

Growth rate of income A	Growth rate of income B	ρ_A	ρ_B	r
- 2 %	2 %	5 %	6 %	5, 5 %

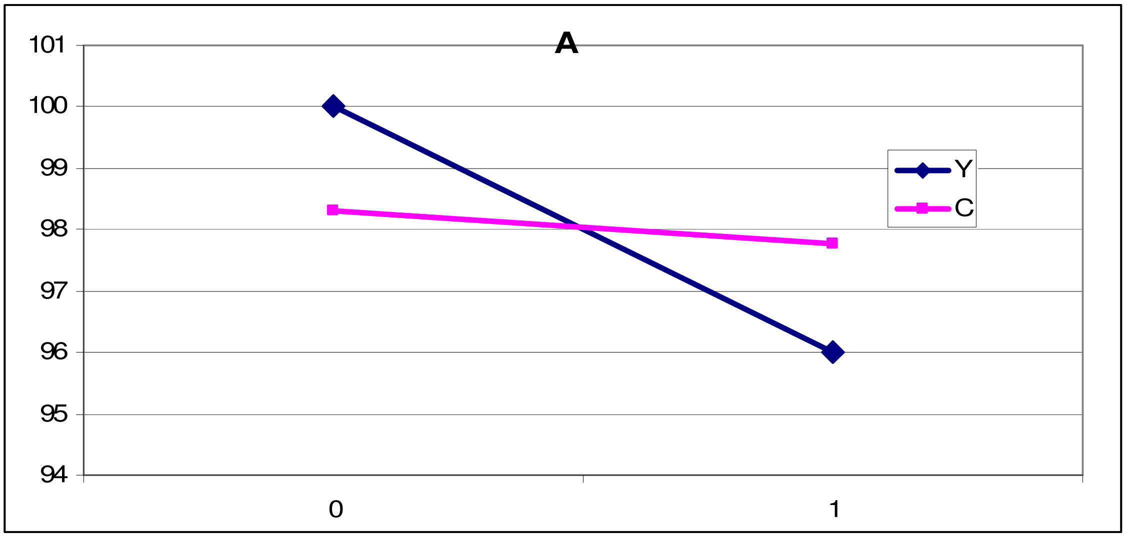


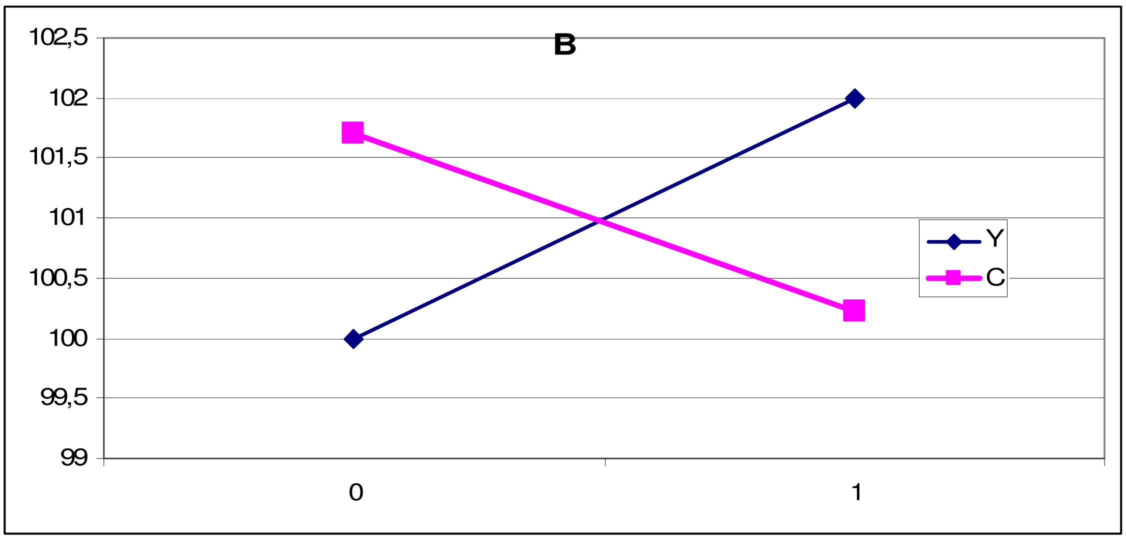


Initial income is the same for both individuals. Income stream of A is decreasing, of B it is increasing.
r is between ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is decreasing ($r<\rho_B$) Individual A is a lender, B is a borrower.
This situation corresponds to Figure no. 32 in the main text.

2)

Growth rate of income A	Growth rate of income B	ρ_A	ρ_B	r
- 4 %	2 %	5 %	6 %	4, 45 %

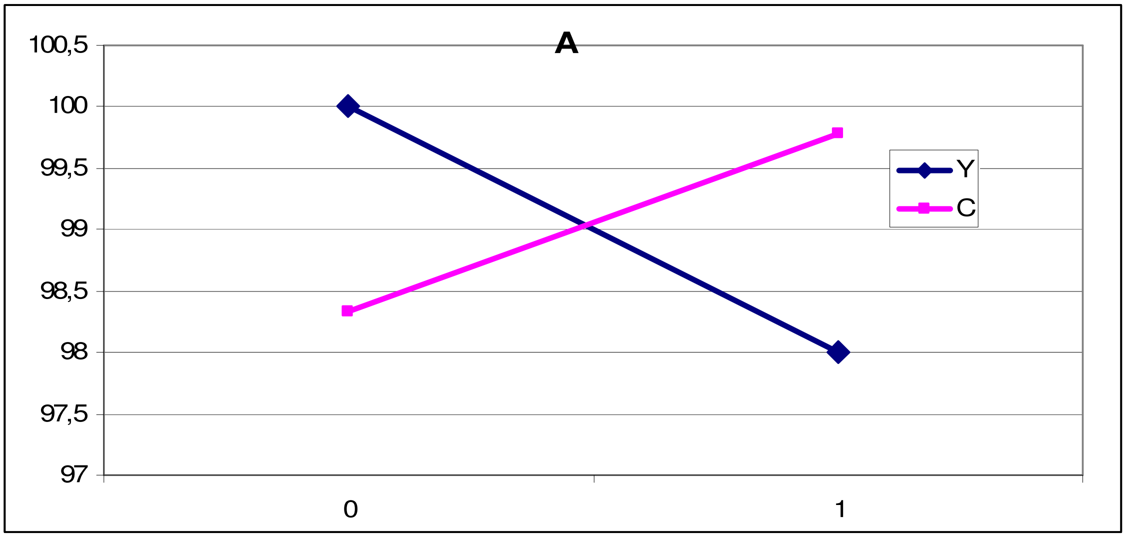


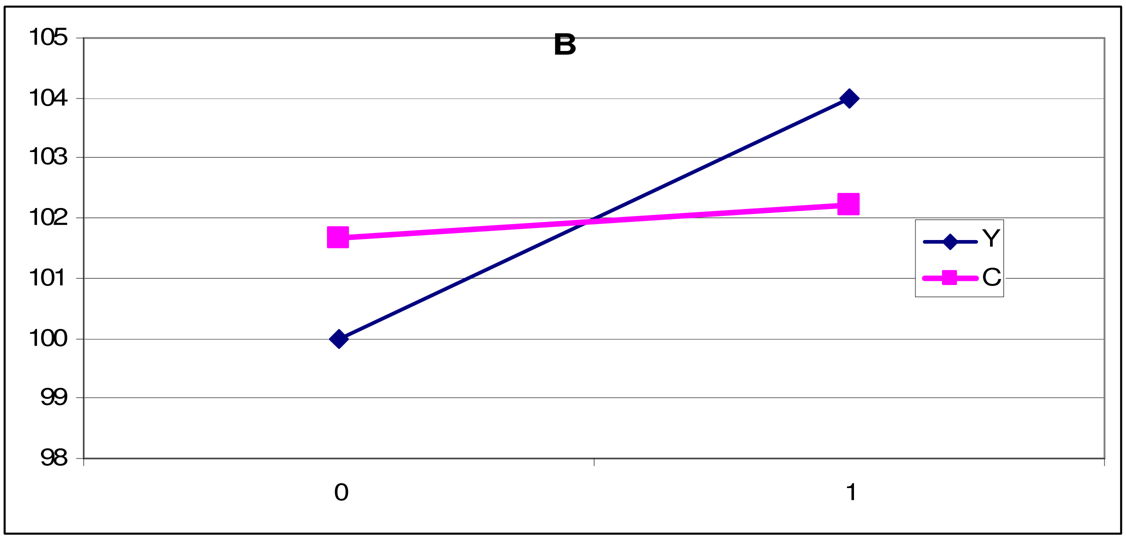


Initial income is the same for both individuals. Income stream of A is decreasing more than income of B is increasing.
r is below ρ_A and ρ_B.
Consumption flow of A is decreasing ($r<\rho_A$), of B it is decreasing ($r<\rho_B$) too. Individual A is a lender, B is a borrower.

3)

Growth rate of income A	**Growth rate of income B**	ρ_A	ρ_B	**r**
- 2 %	**4 %**	**5 %**	**6 %**	**6, 56 %**

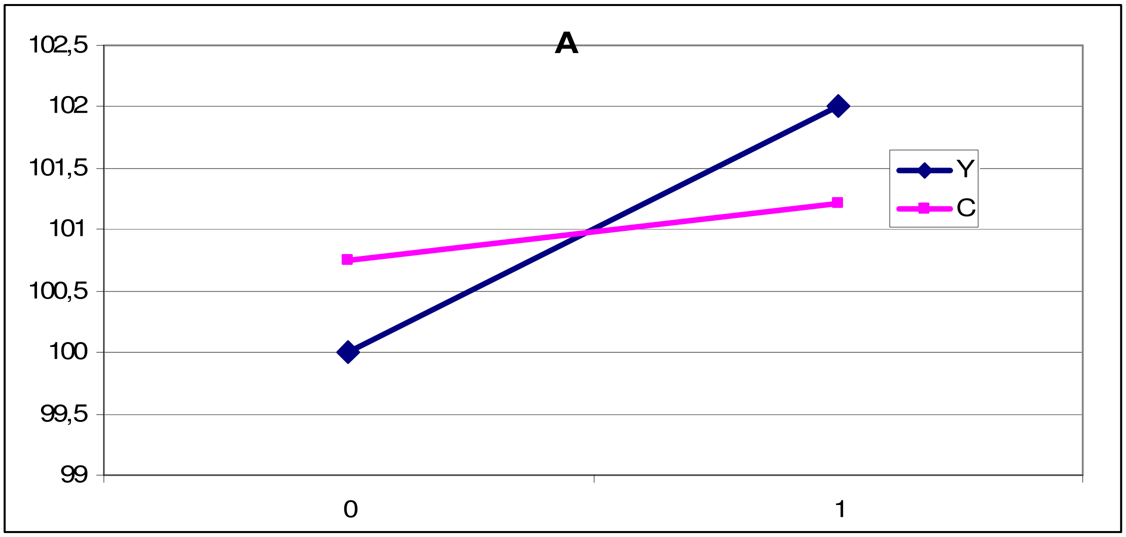


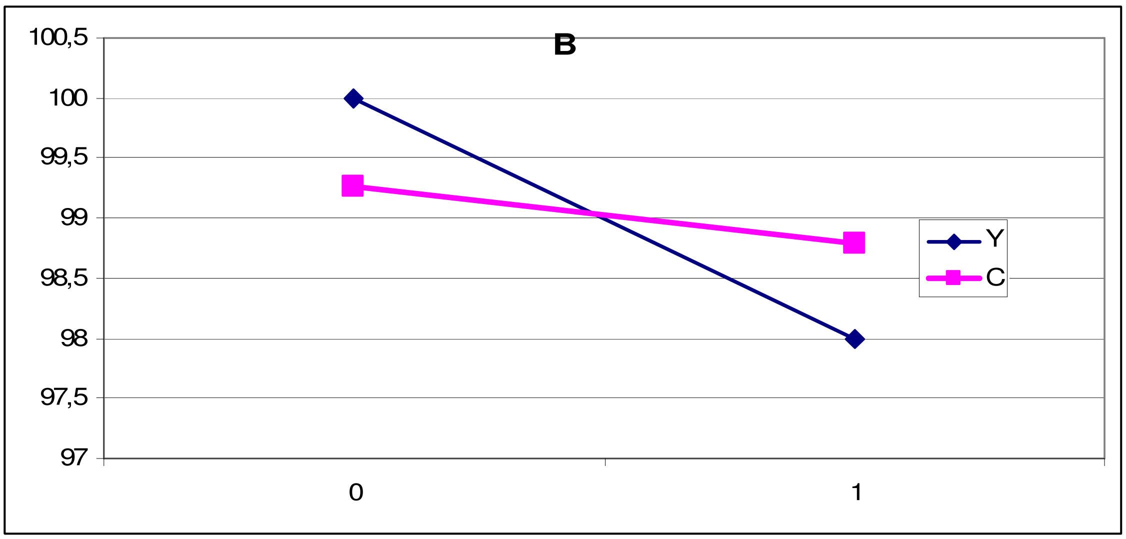


Initial income is the same for both individuals. Income stream of A is decreasing less than income of B is increasing.
r is above ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is increasing ($r>\rho_B$) too. Individual A is a lender, B is a borrower.

4)

Growth rate of income A	Growth rate of income B	ρ_A	ρ_B	r
2 %	-2 %	5 %	6 %	5, 5 %

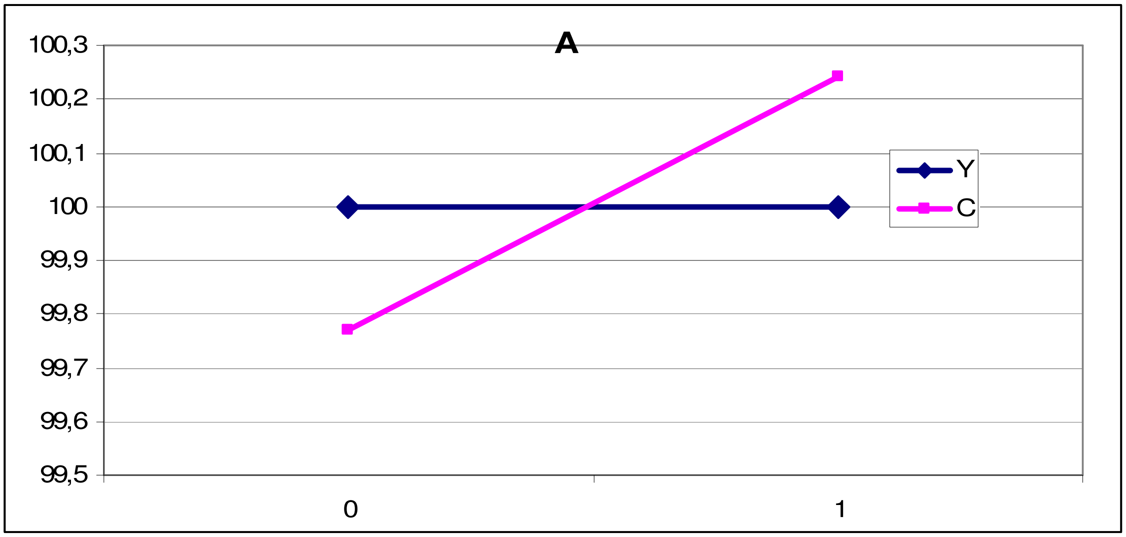


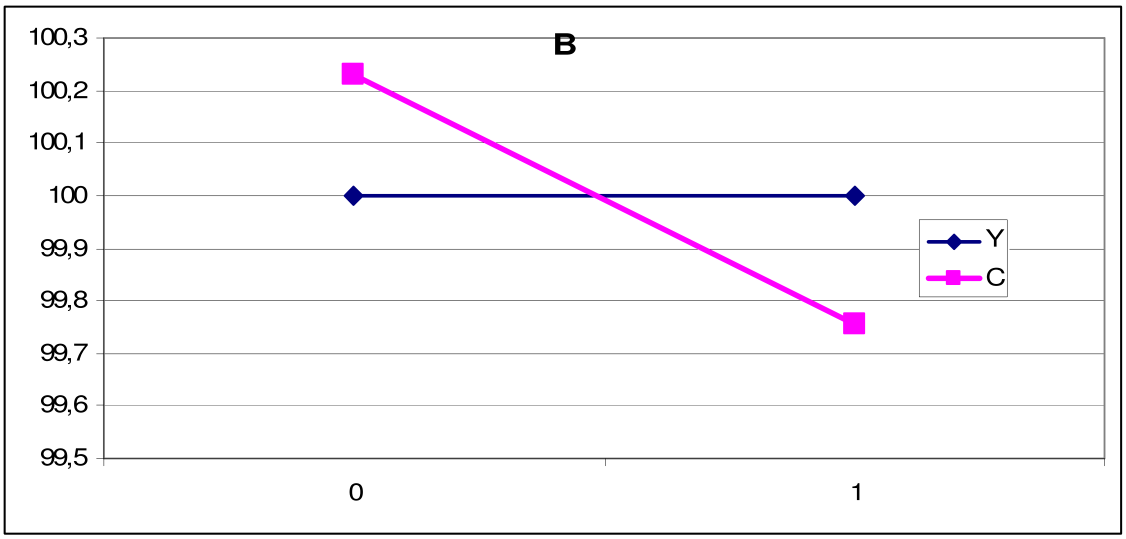


Initial income is the same for both individuals. Income stream of A is increasing, of B is decreasing.
r is between ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is decreasing ($r<\rho_B$). Individual A is a borrower, B is a lender.

5)

Growth rate of income A	**Growth rate of income B**	ρ_A	ρ_B	**r**
0 %	**0 %**	**5 %**	**6 %**	**5, 5 %**

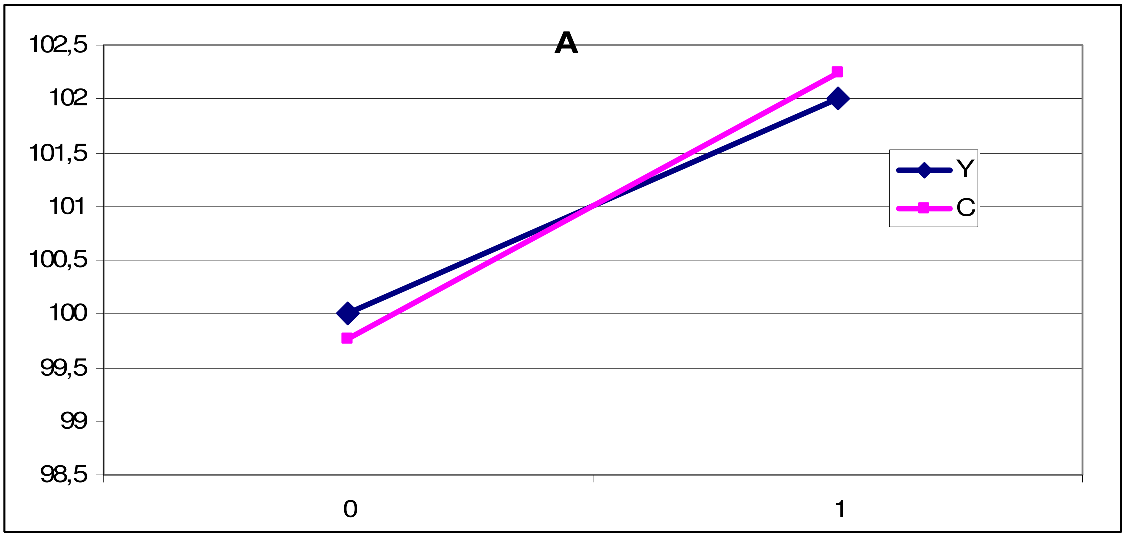


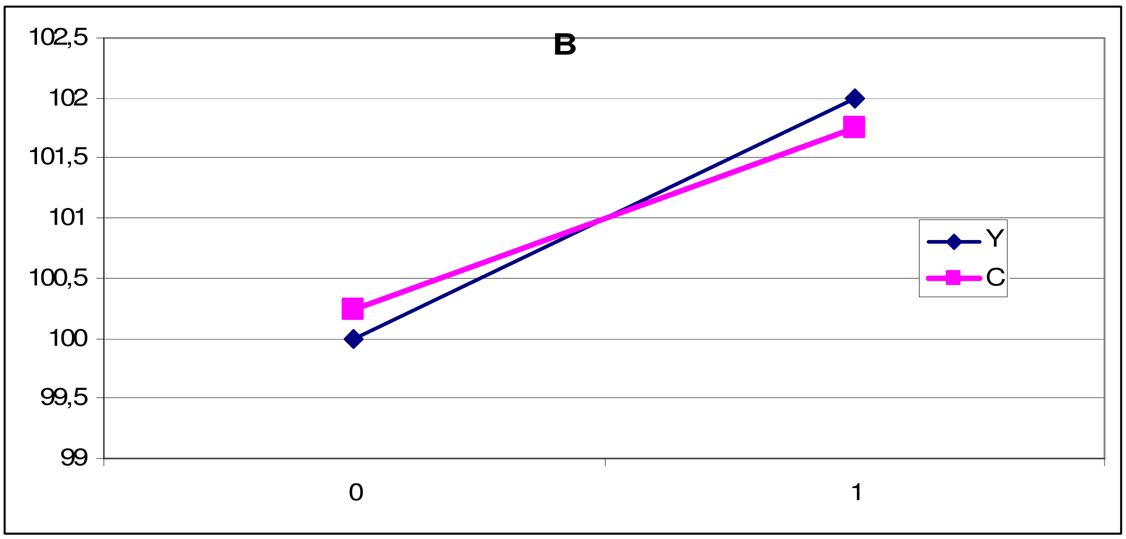


Initial income is the same for both individuals. Income streams of A and B are constant.
r is between ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is decreasing ($r<\rho_B$). Individual A is a lender, B is a borrower.
This situation corresponds to Figure no. 31 in the main text.

6)

Growth rate of income A	**Growth rate of income B**	ρ_A	ρ_B	**r**
2 %	**2 %**	**5 %**	**6 %**	**7, 6 %**

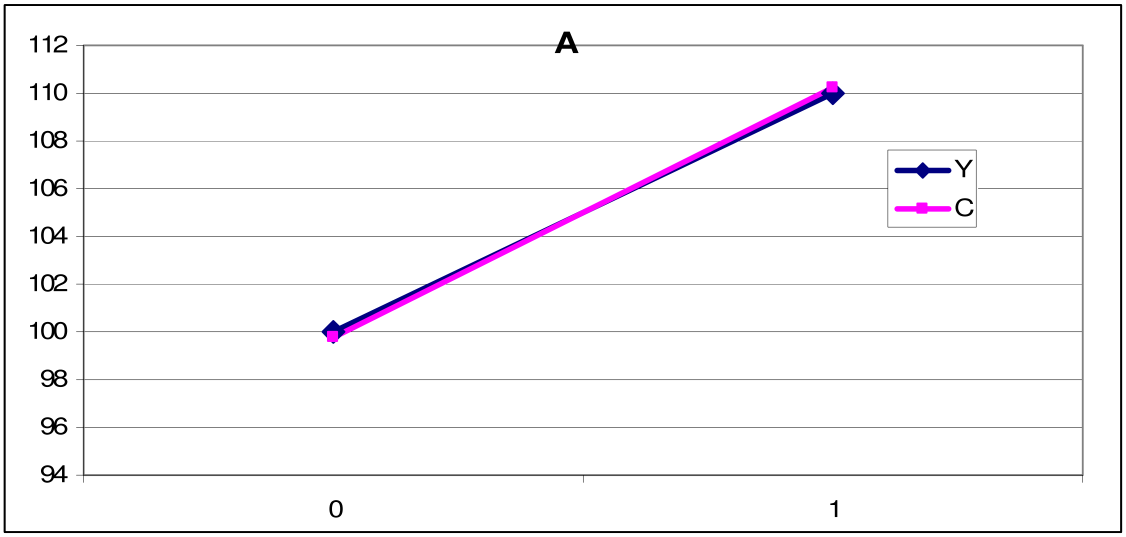


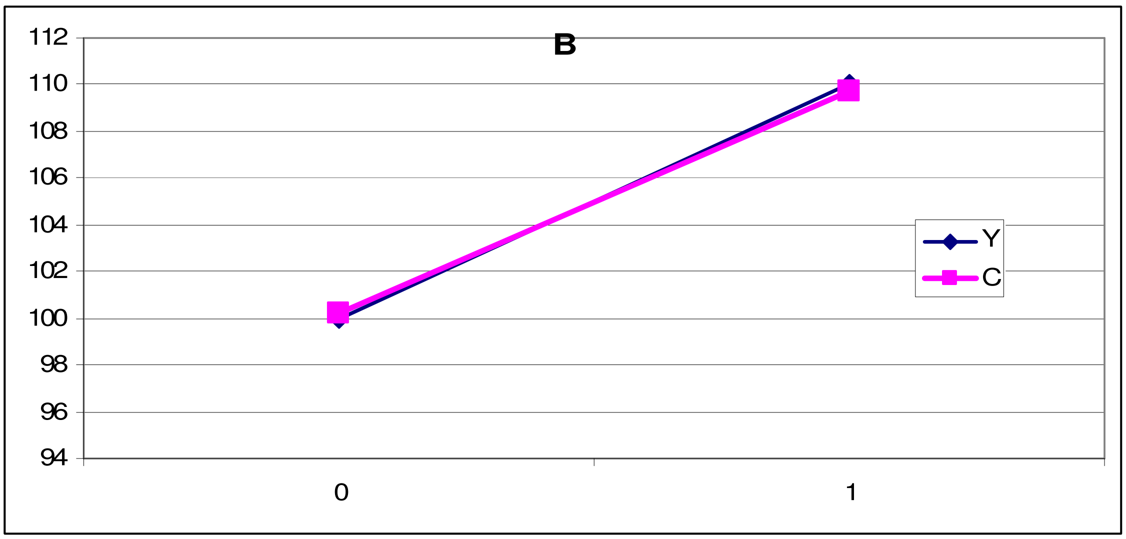


Initial income is the same for both individuals. Income streams of A and B are increasing at the same rate.
r is higher than ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is increasing ($r>\rho_B$) too. Individual A is a lender, B is a borrower.

???)

Growth rate of income A	Growth rate of income B	ρ_A	ρ_B	r
10 %	10 %	5 %	6 %	16 %

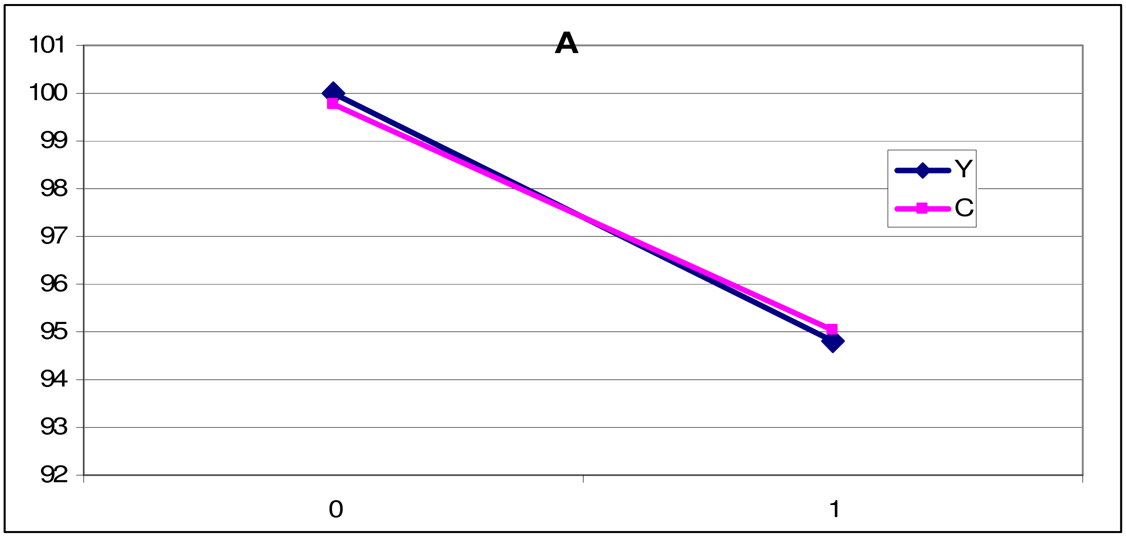


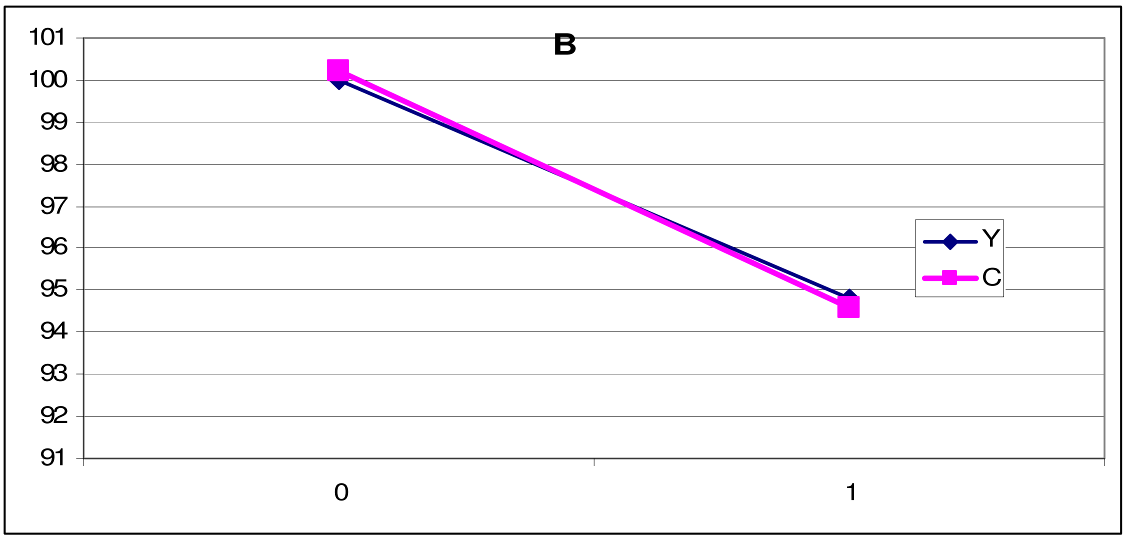


Initial income is the same for both individuals. Income streams of A and B are increasing at the same and a very high rate.
r is higher than ρ_A and ρ_B.
Consumption flow of A is increasing ($r>\rho_A$), of B it is increasing ($r>\rho_B$) too. Individual A is a lender, B is a borrower, but both positions are very close to zero.

7)

Growth rate of income A	**Growth rate of income B**	ρ_A	ρ_B	**r**
-5,21 %	**-5,21 %**	**5 %**	**6 %**	**0 %**





Initial income is the same for both individuals. Income streams of A and B are decreasing at the same rate. This rate is chosen intentionally to reach:

r = 0 %

Consumption flow of A is decreasing ($r<\rho_A$), of B it is decreasing ($r<\rho_B$) too. Individual A is a lender, B is a borrower. Thus, there exists an intertemporal market, an exchange of present goods for future goods.

Appendix 3 UF with leisure time

A) In this appendix, we will add the assumption that people enjoy also their leisure time, not only consumption. Furthermore, we will explicitly assume that the only source of income is their labour income. We will develop a similar two period model as was presented in the main text. In the first version of this model, the phenomenon of the intertemporal substitution of labour will generate some kind of the PPF curve displayed in the main text. However, as we will see, even this curve will depend on utility (or rather disutility), not technical productivity. We will also see that the subjective discount rate will affect not only the shape of the (intertemporal consumption) indifference curves, but also the position of the endowment point(s). Thus, all important outcomes in this sub-model of the theory of interest will depend solely on subjective phenomena.

Consider a representative consumer maximizing his life-time utility in a simple two-period model (Equation 1). For simplicity, assume that $\theta=1$, hence the utility function is logarithmic. His utility depends on consumption C and leisure time H in both periods. Future utilities are discounted by subjective discount rate ρ. The relative weight of consumption and leisure in the utility function is represented by parameter b. This parameter might also play a role that distinguishes discounting of utility from consumption and from leisure. Alternatively, both terms might be discounted by a different subjective discount rate (i.e. ρ_C and ρ_H).[20]

Equation (2) represents his intertemporal budget constraint. W_0 and W_1 stand for real wage earned exogenously in period one and two, respectively. In the second version of this model (B), we will relax the assumption of constant real wage and we explicitly add a production function in both periods that exhibits diminishing marginal product of labour. Alternatively, W_0 and W_1 might be understood as parameters in linear production function $Y_t=A_tL_t$, i.e. $W_0 = dY_0/dL_0 = A_0$ and $W_1 = dY_1/dL_1 = A_1$. Furthermore, labour might be used only in short production processes, i.e. it may be used only in the creation of the given period output (in earning the given period income).

$$U = \ln C_0 + b\ln H_0 + \frac{1}{1+\rho}\ln C_1 + \frac{b}{1+\rho}\ln H_1 \qquad (1)$$

$$C_0 + \frac{1}{1+r}C_1 = \mathrm{W}_0\mathrm{L}_0 + \frac{1}{1+r}\mathrm{W}_1\mathrm{L}_1 \qquad (2)$$

The time constraint in both periods is given by:

$$L_0 + H_0 = 1 \qquad (3)$$

$$L_1 + H_1 = 1 \qquad (4)$$

We normalized the time endowment to 1. Thus, time spent by working (L) and relaxing (H) gives 1 altogether. Substituting (3) and (4) into (1), the lifetime utility function might be written as:

[20] Model from Romer + reasons for different discounting

$$U = \ln C_0 + b\ln(1-L_0) + \frac{1}{1+\rho}\ln C_1 + \frac{b}{1+\rho}\ln(1-L_1) \tag{5}$$

Set up a simple Lagrangian function and solve for the first order conditions (FOC).

$$\mathrm{L} = \ln C_0 + b\ln(1-L_0) + \frac{1}{1+\rho}\ln C_1 + \frac{b}{1+\rho}\ln(1-L_1) + \lambda(\mathrm{W}_0\mathrm{L}_0 + \frac{1}{1+r}\mathrm{W}_1\mathrm{L}_1 - C_0 - \frac{1}{1+r}C_1) \tag{6}$$

FOCs for consumption are:

$$\frac{\partial L}{\partial C_0} = \frac{1}{C_0} - \lambda = 0 \tag{7}$$

$$\frac{\partial L}{\partial C_1} = \frac{1}{1+\rho}\frac{1}{C_1} - \lambda\frac{1}{1+r} = 0 \tag{8}$$

(7) and (8) imply:

$$\frac{C_1}{C_0} = \frac{1+r}{1+\rho} \tag{9}$$

Equation (9) represents the Euler (consumption) equation for this problem.

FOCs for labour are given by:

$$\frac{\partial L}{\partial L_0} = \frac{-b}{1-L_0} + \lambda W_0 = 0 \tag{10}$$

$$\frac{\partial L}{\partial L_1} = \frac{1}{1+\rho}\frac{-b}{1-L_1} + \lambda\frac{W_1}{1+r} = 0 \tag{11}$$

(10) and (11) imply:

$$\frac{1-L_1}{1-L_0} = \frac{W_0}{W_1}\frac{1+r}{1+\rho} \tag{12}$$

Equation (12) represents the Euler (employment) equation for this problem. It describes the optimal allocation of leisure (labour) over time. This system has five unknowns (C_0, C_1, L_0, L_1,λ) in five equations (7,8,10,11,2). Leisure time can be then easily determined from time constraints (3) and (4).

(7) and (10) imply:

$$\frac{1}{C_0} = \frac{b}{1-L_0}\frac{1}{W_0} \tag{13}$$

This problem might be more easily solved for the leisure time. Hence (13) becomes:

$$\frac{1}{C_0} = \frac{b}{H_0}\frac{1}{W_0} \qquad (14)$$

Similar manipulations can be done with (8) and (11), which yields:

$$\frac{1}{C_1} = \frac{b}{H_1}\frac{1}{W_1} \qquad (15)$$

Substituting (14) and (15) into (2) and using time constraints (3) and (4), equation (2) becomes:

$$\frac{W_0}{b}H_0 + \frac{W_1}{b(1+r)}H_1 = \mathrm{W}_0(1-\mathrm{H}_0) + \frac{1}{1+r}\mathrm{W}_1(1-\mathrm{H}_1) \qquad (16)$$

Using (12), equation (16) takes the form:

$$\frac{W_0}{b}H_0 + \frac{W_0}{b(1+\rho)}H_0 = \mathrm{W}_0 - \mathrm{W}_0\mathrm{H}_0 + \frac{1}{1+r}\mathrm{W}_1 - \frac{\mathrm{W}_0}{1+\rho}\mathrm{H}_0 \qquad (17)$$

A simple (but time-consuming) rearrangement of terms above gives us:

$$H_0\left[\frac{W_0}{b} + \mathrm{W}_0 + \frac{W_0}{b(1+\rho)} + \frac{\mathrm{W}_0}{1+\rho}\right] = \mathrm{W}_0 + \frac{1}{1+r}\mathrm{W}_1 \qquad (18)$$

$$H_0^* = \frac{1 + \frac{1}{1+r}\frac{\mathrm{W}_1}{\mathrm{W}_0}}{1 + \frac{1}{b} + \frac{1}{1+\rho} + \frac{1}{b(1+\rho)}} \qquad (19)$$

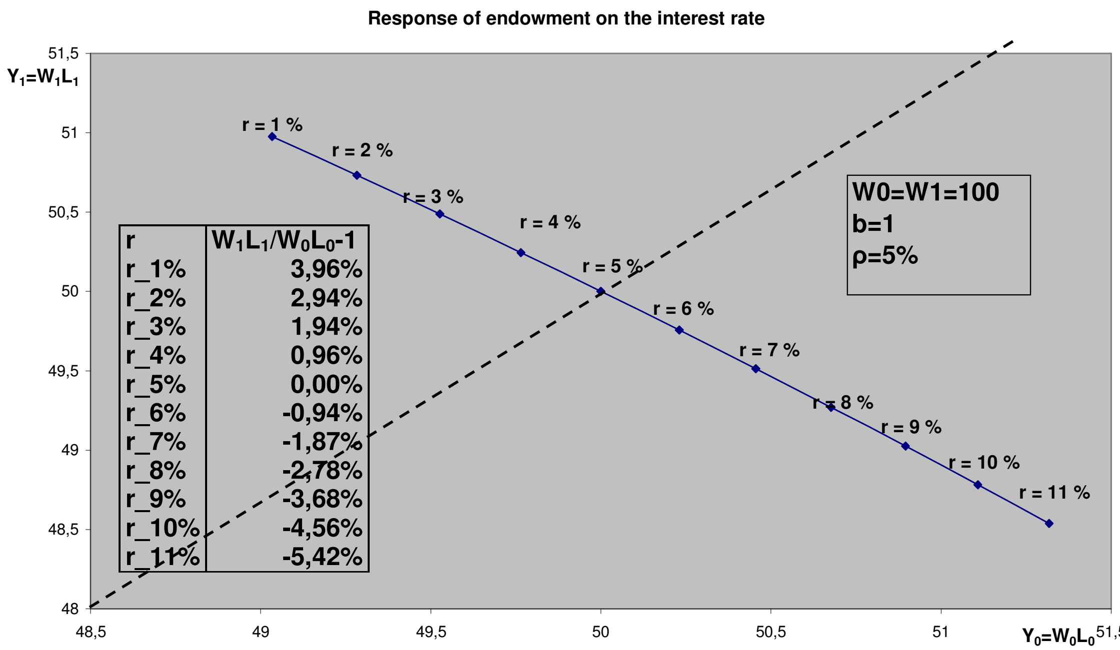
Optimum H_1 is given by (19) and (12):

$$H_1 = \frac{\mathrm{W}_0}{\mathrm{W}_1}\frac{1+r}{1+\rho} \times \frac{1 + \frac{1}{1+r}\frac{\mathrm{W}_1}{\mathrm{W}_0}}{1 + \frac{1}{b} + \frac{1}{1+\rho} + \frac{1}{b(1+\rho)}} \qquad (20)$$

$$H_1^* = \frac{1 + (1+r)\frac{\mathrm{W}_0}{\mathrm{W}_1}}{(2+\rho) + \frac{1+\rho}{b} + \frac{1}{b}} \qquad (21)$$

Thus, leisure time (labour) in the present increases (decreases) and leisure time (labour) in the future decreases (increases), if the interest rate falls, the relative intertemporal wage W_1/W_0 rises, or if the subjective discount rate grows. As a result, for the given interest rate, higher impatience (ρ) moves the endowment point closer to the vertical axis in the C_1-C_0 space and further from the horizontal axis, because present labour supply (and therefore labour income) falls and future rises. This represents another channel that might raise the interest rate in equilibrium after the increase in ρ, because the time shape of the income stream will become even more increasing.

If we realise that $L^* = (1-H^*)$ in every period, the total labour income in the given period is $Y_t=W_tL_t^*$. The impact of various rates of interest (for given ρ) on the endowment point is portrayed in the graph below.



As can be seen, for a constant wage over time the income stream is smoothed, if the interest rate is equal to the subjective discount rate. This picture closely resembles a usual PPF curve presented in the main text. Lower interest rate moves the endowment closer to the vertical axis. However, there is no element of productivity in our analysis. This PPF depends only on the utility of leisure time. As a result, the equilibrium real rate of interest will also depend only on subjective phenomena.

A decrease in the interest rate and the resulting change in the budget line of an individual are presented in Figure no. 1_A3 below. As can be seen, lower interest rate moves the income endowment point closer to the vertical axis from A^1 to A^2. At the same time, the budget line becomes flatter. In standard analysis, the budget line rotates around the endowment point A. Here, however, the pivot point itself is being moved.[21]

It is obvious that an increase in the interest rate decreases the growth rate in income over time, because it is more profitable to work in the present and relax in the future. Thus, at the individual level we found an inverse relationship between the interest rate and the shape of the income stream. The phenomenon of the intertemporal substitution of labour introduces a new channel that partly offsets the impact of the income stream on the interest rate presented in

[21] In supplement 2 to this Appendix, we show that the endowment point A^2 is below the old budget line, if (for a constant stream of wages) the interest rate is lower than the subjective discount rate (i.e. $r < \rho$) and vice versa.

our previous analysis. To find the ultimate impact on the interest rate, we have to analyse the optimum consumption stream in this model.[22]

By substituting Euler consumption equation (9) and equations (14) and (15) into the intertemporal budget constraint (2), we get:

$$C_0 + \frac{1}{1+\rho}C_0 = \mathrm{W}_0(1 - \frac{bC_0}{\mathrm{W}_0}) + \frac{\mathrm{W}_1}{1+r}(1 - \frac{\mathrm{b}}{\mathrm{W}_1}\frac{1+\mathrm{r}}{1+\rho}C_0) \tag{22}$$

$$\frac{(1+\rho)C_0 + C_0}{1+\rho} = \mathrm{W}_0 - bC_0 + \frac{\mathrm{W}_1}{1+r} - \frac{\mathrm{b}}{1+\rho}C_0 \tag{23}$$

$$\frac{(1+\rho)C_0 + C_0 + b(1+\rho)C_0 + bC_0}{1+\rho} = \mathrm{W}_0 + \frac{\mathrm{W}_1}{1+r} \tag{24}$$

$$C_0(2+\rho+2b+b\rho) = \frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(1+r)}(1+\rho) \tag{25}$$

$$C_0(2+\rho)(1+b) = \frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(1+r)}(1+\rho) \tag{26}$$

$$C_0(2+\rho)(1+b) = \frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(1+r)}(1+\rho) \tag{27}$$

$$C_0^* = \frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(1+r)}\frac{(1+\rho)}{(2+\rho)(1+b)} \tag{28}$$

If b=0, this expression perfectly coincides with equation (7) in Appendix 2. Optimum future consumption is derived, if we substitute (28) into (9):

$$C_1^* = \frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(2+\rho)(1+b)} \tag{29}$$

[22] However, one possible (and most probably correct) interpretation is as follows: Reduction in the interest rate leads to a shift of the optimal point which represents the ideal intertemporal allocation of labour and the resulting income endowment (along a hypothetical PPF) to the left (i.e. the growth rate in income rises). At the same time, a decline in the interest rate results in a decrease in the optimal growth rate of consumption. Thus, the equilibrium interest rate can be found where these two tendencies offset each other. On the PPF, an increasing income stream is consistent with lower interest rate. In case of the (consumption) indifference curve, increasing consumption stream is associated with higher interest rate. Thus, it can be said that lower interest rate decreases the supply of present goods (due to the reduction in the supply of present labour) and raises the demand for present goods (due to higher consumption demand). An increase in the interest rate has the opposite diverging effects. Thus, the interest rate must adjust to equilibrate these two tendencies.

As is perfectly clear from (28) and (29) parameter b (preference for leisure time) decreases consumption in both periods. Furthermore, comparing C_0^* and $W_0L_0^*$, we can determine whether the given individual is a lender or a borrower for the given r. Figure no. 1_A3 below shows a consumer, whose subjective discount rate is higher than the real interest rate. With regard to the stream of wages, it is either increasing ($W_1>W_0$) or constant. As a result, according to (12) his endowment point is above the 45º line, because he works relatively more in the future. Thus, he also earns relatively more in the future ($Y_1=W_1L_1^*>Y_0=W_0L_0^*$). At the same time, according to (9) his optimal consumption stream is decreasing ($C_1^*< C_0^*$). This particular consumer is a borrower, because $C_0^*> Y_0=W_0L_0^*$.

Now, consider a reduction in the interest rate ($r_2< r_1$). As can be seen in Figure no. 1_A3, it will raise present consumption from C_0^{1*} to C_0^{2*} and reduce present labour supply (and increase present leisure), which will consequently decrease present labour income from Y_0^1 to Y_0^2. Decline in the interest rate is beneficial for a debtor, as the new optimum is posited at a higher indifference curve.[23] Moreover, a reduction in the interest rate decreases the amount of saving to a greater extent, if the intertemporal substitution of labour exists compared with its absence. The reason lies in a decline in present income endowment, which drives up the difference between present income (Y_0) and present optimal consumption (C_0*).

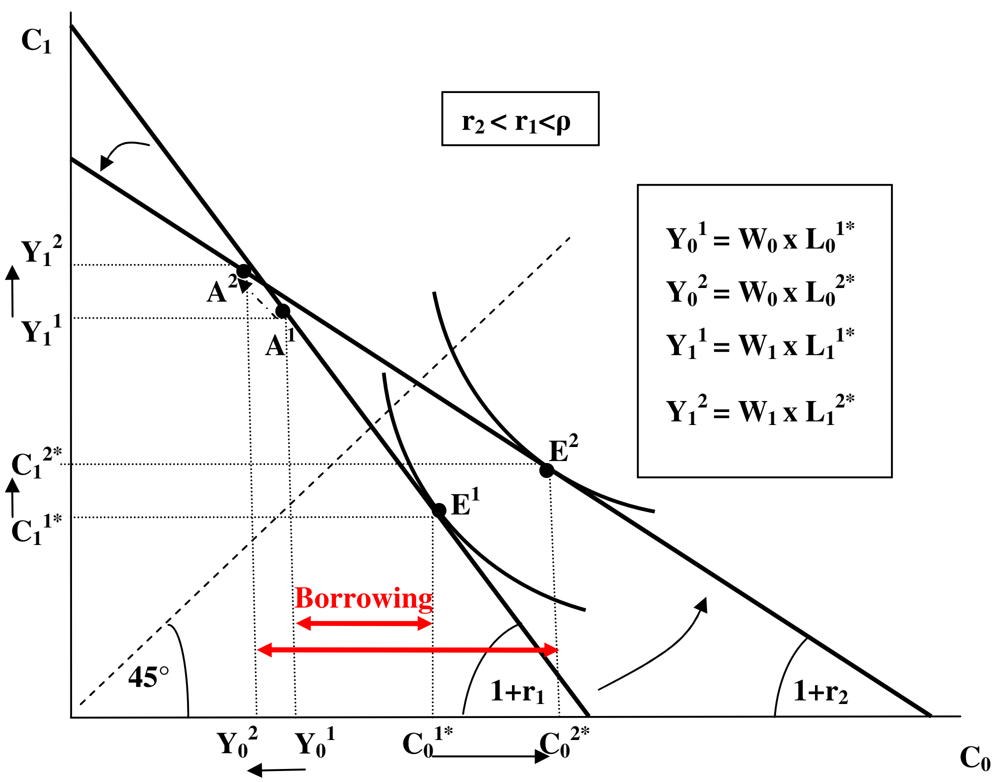


Figure no. 1_A3. An impact of a decrease in the interest rate on consumption and on income endowment, if the intertemporal substitution of labour is effective.

Thus, the saving curve is more elastic, if the intertemporal substitution in labour (ISL) is included in the model. The reason lies in the fact that the reduction in the interest rate

[23] Even though this indifference curve represents utility only from consumption and not from leisure, our conclusion seems to correct, because present leisure increases, even though the future leisure falls. Furthermore, the increase in consumption in both periods due to the reduction in the interest rate will surely benefit the debtor.

decreases present labour supply and hence the present labour income and raises future labour supply and future labour income. Both changes in income shift the traditional saving curve to the left. As a result, the saving curve that includes both the intertemporal substitution in consumption (ISC) and in labour might be constructed as follows: A drop in the interest rate moves the optimum saving along the traditional saving curve, which neglects ISL, from point E^1 to point B. The second round effect on the income stream shifts the entire traditional saving curve to the left. The new point of optimum can be found at point E^2. Connecting points E^1 and E^2, the more general saving curve can be found (S^{ISL}). This curve reflects both the ISC and the ISL. As can be seen, our representative consumer makes negative saving, since present consumption exceeds present income.[24]

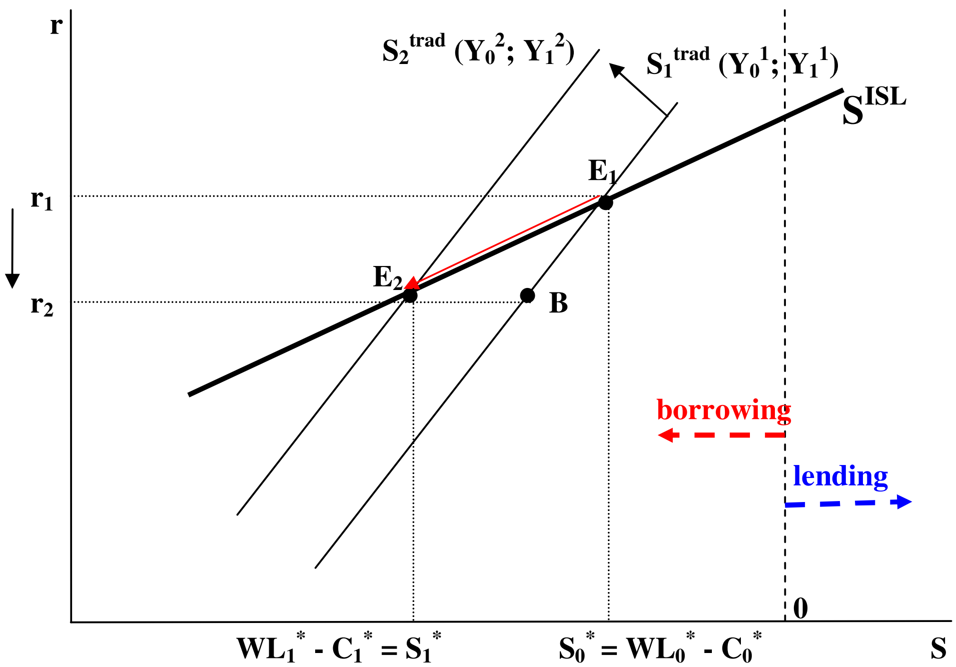


Figure no. 1B_A3. Construction of the saving curve, which includes both the intertemporal substitution in consumption and in labour.

Furthermore, if we relax the assumption that the subjective discount rate and the stream of wages are the same for all individuals, we can derive the equilibrium real interest rate for a general intertemporal model. The aggregate constraint for such an economy would be analogous to (9) and (10) from Appendix 2:

$$C_0^{A*} + C_0^{B*} = W_0^A L_0^{A*} + W_0^B L_0^{B*} \qquad (30)$$

[24] Linear saving curve is constructed just for simplicity. As can be seen from (19) and (28), the relationship between optimum saving and real interest rate must be clearly non-linear. Furthermore, logarithmic utility function and the presence of future labour income leads to an upward sloping saving curve. The response of present consumption to the change in the interest rate is negative (see equation 28): $\partial C_0^*/\partial r = -K.W_1/(1+r)^2$, where $K = (1+\rho)/[(2+\rho)(1+b)]$. Thus, with lower interest rate, optimum saving declines. If there was no future wage ($W_1 = 0$), the saving curve would be vertical (neither optimum present consumption, nor optimum present leisure would depend on the interest rate).

$$C_1^{A*} + C_1^{B*} = \mathrm{W}_1^{\mathrm{A}} \mathrm{L}_1^{\mathrm{A}*} + \mathrm{W}_1^{\mathrm{B}} \mathrm{L}_1^{\mathrm{B}*} \qquad (31)$$

However, such an analysis would be too complicated compared with the results acquired, since these would not surpass those already discussed in Appendix 2. Thus, let us assume that all individuals are identical with regard to their subjective discount rate and their exogenous stream of wages. Such homogeneity implies that individual saving is zero on the part of each individual.

As a result, neither the interest rate r_1 nor r_2 in Figure no. 1_A3 is a good candidate for an equilibrium rate of interest. Both are too low, since they result in the excess of demand for present goods over their available supply ($C_0^* > Y_0$). The condition for an equilibrium rate of interest is thus given by:

$$C_0^* = \mathrm{W}_0 \mathrm{L}_0^* = Y_0 \qquad (32)$$

From (28), (19) and (3) we get:

$$\frac{\mathrm{W}_0(1+r)+\mathrm{W}_1}{(1+r)} \frac{(1+\rho)}{(2+\rho)(1+b)} = \mathrm{W}_0(1 - \frac{1+\frac{1}{1+r}\frac{\mathrm{W}_1}{\mathrm{W}_0}}{1+\frac{1}{b}+\frac{1}{1+\rho}+\frac{1}{b(1+\rho)}}) \qquad (33)$$

The only unknown is the real interest rate r. However, instead of solving (33) we can directly substitute (32) into (13) and a similar constraint $C_1^* = \mathrm{W}_1 \mathrm{L}_1^*$ into (15), which yields:

$$L_0 = \frac{1}{1+b} \qquad (34)$$

$$L_1 = \frac{1}{1+b} \qquad (35)$$

Thus, the labour supply will be the same in both periods. As a result, the equilibrium interest rate will depend only on the flow of wages and the subjective discount rate. Substitute (34) and (35) into (12):

$$(1+r) = \frac{W_1}{\mathrm{W}_0}(1+\rho) \qquad (36)$$

(36) closely resembles equation (30) from Appendix 2. There is therefore no need to repeat the analysis again. If the flow of wages is constant, the equilibrium real interest rate will be equal to the subjective discount rate. If the stream of wages is increasing, the interest rate will be greater than the subjective discount rate.

However, in this particular case the equilibrium income endowment will not be affected by the intertemporal substitution of labour. The reason is as follows: An increase in the average intertemporal wage (W_1/W_0) will benefit present leisure time. However, higher W_1/W_0 will accordingly increase the real interest rate, which perfectly offsets the original tendency. Hence, the equilibrium of a representative individual might be represented by Figures 29 or 30 in the main text. The endowment point and the resulting equilibrium interest rate will depend only on the time shape of wages, not on the allocation of labour over time, since it is constant. Moreover, parameter b (the relative importance of leisure in the utility function) does not affect the equilibrium interest rate either.

In other words, in this homogenous-agent model the intertemporal allocation of labour will not be affected by the time shape of the stream of wages, because any shape will accordingly modify the equilibrium interest rate, which will eventually leave the optimal intertemporal allocation of labour at the previous level that is characterised by L=1/(1+b) in every period.

A similar analysis can be done for the subjective discount rate. Its rise will increase the real interest rate by the same amount keeping the equilibrium intertemporal allocation of labour unaffected. The only outcome will be a steeper indifference curve and a steeper budget line. There will be no impact on the representative endowment point.

This analysis is presented in Figure no. 2_A3 below. We assume a constant stream of wages ($W_1=W_0=W$). The labour supply in both periods is the same $L^*=1/(1+b)$. As a result, the time shape of the income stream is constant ($Y_0=Y_1=Y=WxL^*$). According to equation (36), the interest rate must be equal to the subjective discount rate. Thus, consumption is also smoothed over time (see equation 9). Now, consider an increase in the subjective discount rate. This will benefit present leisure time at the expense of future leisure time, so present labour falls and future labour increases. As a result, present income decreases from Y to $Y_0=WL_0^*$ and consequently the income endowment point moves from A^1 to A^2 (see panel a).[25] At the same time, the indifference curve will become steeper (a similar analysis was made in Appendix 2). The resulting excess of demand for present goods over their supply ($C_0^{2*}>Y_0=WL_0^*$) is greater compared with the situation if leisure is not included in the utility function (compare the size of borrowing represented by the red solid line and the dashed line). The reason is that the intertemporal substitution of labour moves the income endowment point to the top left. Yet, to equilibrate the demand for present goods (C_0^*) with their available supply ($Y=W_0L_0^*$) the interest rate must go up. In the end, the interest rate is equal to the new subjective discount rate ($r_2 = \rho_2$). Furthermore, labour supply is the same in both periods. The same holds for income and consumption. Thus, the endowment point is eventually in the same position as it was in the beginning (see panel b).

[25] In supplement 1 at the end of this section it is proved that the new endowment point must lie on the original budget line because a change in the subjective discount rate leaves the present value of the income stream unaffected. Alternatively, it is demonstrated that the relative shift of the endowment point is in the direction of (1+r), which perfectly coincides with the slope (in absolute value) of the intertemporal budget constraint.

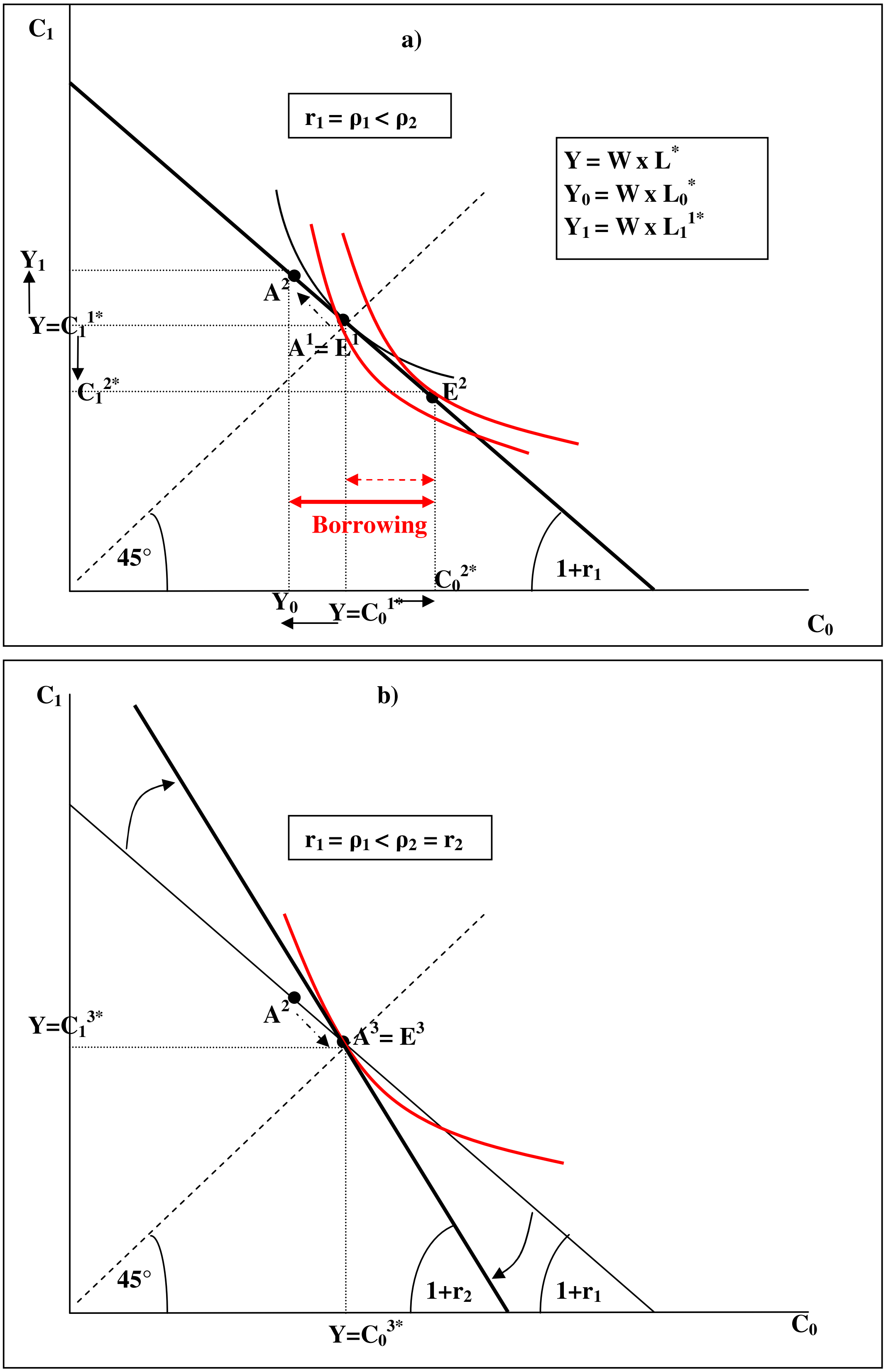


Figure no. 2_A3. An impact of an increase in the subjective discount rate on consumption, income endowment and eventually on the interest rate

However, the individual intertemporal substitution of labour might play an important role in equilibrium if there is heterogeneity across individuals. Equation (33) for heterogeneous agents would be modified to (see equation 30):

$$\frac{W_0^A(1+r)+W_1^A}{(1+r)}\frac{(1+\rho_A)}{(2+\rho_A)(1+b)}+\frac{W_0^B(1+r)+W_1^B}{(1+r)}\frac{(1+\rho_B)}{(2+\rho_B)(1+b)}=$$

$$=W_0^A\left(1-\frac{1+\frac{1}{1+r}\frac{W_1^A}{W_0^A}}{1+\frac{1}{b}+\frac{1}{1+\rho_A}+\frac{1}{b(1+\rho_A)}}\right)+W_0^B\left(1-\frac{1+\frac{1}{1+r}\frac{W_1^B}{W_0^B}}{1+\frac{1}{b}+\frac{1}{1+\rho_B}+\frac{1}{b(1+\rho_B)}}\right) \qquad (37)$$

We do not have the ambition to solve this complicated equation for the equilibrium interest rate r. Yet, it is obvious that it will depend negatively on present wages W_{0i} and positively on future wages W_{1i} and subjective discount rates ρ_i. Furthermore, more patient people (low ρ_i) will consume less and work more in the present. As a result, their net lending position will be positive. It will be more positive than if the intertemporal substitution of labour does not exist. The opposite result would hold for less patient individuals (high ρ_i). Next, an increasing stream of wages will lead to a lower present labour supply and therefore even to a lower present income. Thus, this channel will further raise borrowing of people with an increasing time shape of wages.

As can be seen, the inclusion of leisure into the utility function and the resulting intertemporal substitution of labour reinforce the results obtained in Appendix 2. The reason lies in the fact that, in the first place, the subjective discount rate influences the position of the individual endowment point (provided that r does not move one-for-one with ρ_i). In the second place, the shape of the exogenous stream of wages affects the position of the endowment point ($Y_{0i} = W_{0i} \times L_{0i}^*; Y_{1i} = W_{1i} \times L_{1i}^*$) not only directly due to the magnitude of W_{1i}/W_{0i}, but also due the impact on the optimal allocation of labour (L_{1i}^*; L_{0i}^*). Thus, both the subjective discount rate and the exogenous flow of wages will in turn affect the individual net borrowing/lending position and maybe the resulting equilibrium interest rate.[26] In other words, each individual exogenous parameter might have a stronger impact on the equilibrium interest rate, if the intertemporal substitution of labour exists.

Pure time preference theorists have never discussed the possibility of the intertemporal substitution of labour. Yet, this channel might amplify the link between the time preference (in sense two) and the natural rate of interest. The reason is that time preference favours not only present satisfaction from consumption goods, but it also favours present leisure. As a result, relatively greater leisure time in the present (and lower in the future) reduces the provision of present goods and improves their future provision. This phenomenon therefore supports the first Bohm-Bawerkian ground for interest. It can be said that owing to the preference for present leisure time (and the resulting intertemporal substitution of labour) the second cause for interest reinforces the first cause for interest due to the impact on the relative provision of goods over time.

[26] It seems that the impact on the equilibrium interest rate is not stronger for logarithmic utility function (i.e. for θ=1), but it might be for θ different from 1.

B) In Part A, we assumed a constant wage in each period that is not affected by changes in the labour supply. This assumption is rather strong especially in the general equilibrium model, however, it helped us to focus on specific aspects in the theory of interest. In the present section, we will relax this assumption, as we introduce production function that exhibits diminishing marginal product of labour. Output (and income) will depend on the amount of labour expended in the given period and the real wage will be equal to the marginal product of labour. In this section, labour can be used only in short processes, so present labour might not be used in a longer process that will mature in the next period. This extension will be postponed to section C.

The structure of this model is the same as in section A with only one exception. Output in the present period and in the future period respectively depends on the amount of labour expended in the given period and the level of technologies A_t:

$$Y_0 = A_0 L_0^{\alpha} \quad (38)$$

$$Y_1 = A_1 L_1^{\alpha} \quad (39)$$

The marginal product of labour is decreasing, because $0<\alpha<1$. The intertemporal budget constraint (2) is then modified to:

$$C_0 + \frac{1}{1+r} C_1 = A_0 L_0^{\alpha} + \frac{1}{1+r} A_1 L_1^{\alpha} \quad (40)$$

Furthermore, the time endowment will be generalized to T=L+H. The Euler (employment) equation (12) will result in:

$$\frac{T - L_1}{T - L_0} = \frac{A_1 L_1^{1-\alpha}}{A_0 L_0^{1-\alpha}} \frac{1+r}{1+\rho} \quad (41)$$

Solution of this system will not be presented here, as it seems to be too complicated to be worthwhile. However, the main conclusions from the previous sections are preserved here as well. Present labour supply increases with lower subjective discount rate and higher interest rate. Thus, the PPF curve will be generated again, even though it will depend also on the decreasing marginal productivity of labour (not capital) and its shape will be most probably concave due to this property. As a result, the endowment points will then critically depend on the subjective discount rate (and the level of technologies). Thus, the interest rate in this economy will depend mainly on subjective psychological elements, although the marginal productivity of labour (not capital!) will also affect its size.

C) In the last section, we will only briefly outline a model, in which present labour might be used not only in the production of the given period output, but in which the present labour can be employed also in a longer (and more productive) process that will, however, provide output in the next period. Furthermore, the longer process will also require application of labour in the next period to be fully completed.

The time constraint in each period is given by the following equations:

$$L_0^S + L_0^L + H_0 = T \quad (42)$$

$$L_1^S + L_1^L + H_1 = T \quad (43)$$

We assume that the time endowment in each period is the same T. L_0^S stands for the amount of labour applied in the present in the short production process. L_0^L represents the amount of present labour that is applied in the long process that will mature in the future. L_1^S is applied in the short process in the future, whereas L_1^L is the amount of future labour that is used to finish the output, whose production started in the present.

We will assume that longer processes are more productive. However, some amount of future labour must be employed to finish the longer process. We also allow for a change in the level of technologies over time, hence A_0 need not equal A_1. Furthermore, we assume that technology (knowledge) is non-rival and it might be used in full in both production methods. As a result, the output of consumption goods in each period is given by the following production functions:

$$Y_0 = A_0 (L_0^S)^{\alpha} \qquad (44)$$

$$Y_1^S = A_1 (L_1^S)^{\alpha} \qquad (45)$$

$$Y_1^L = A_1 (L_1^L)^{\beta} (L_0^L)^{\gamma} \qquad (46)$$

The total output of consumable goods in the future is:

$$Y_1 = Y_1^S + Y_1^L \qquad (47)$$

We assume that the marginal product of labour is decreasing, thus α, β, γ are all between 0 and 1. However, it is assumed that longer process is more productive than the shorter process, therefore $\alpha<\beta$ and $\alpha < \gamma$. Furthermore, labour applied in the present in the longer process is more remunerative than labour applied in the future in the same process, hence $\beta < \gamma$. Finally, we assume that the returns to scale in the longer process are not increasing, which means that $\beta + \gamma \leq 1$.

The lifetime utility function (1) is preserved and the intertemporal budget constraint is given by:

$$C_0 + \frac{1}{1+r} C_1 = Y_0 + \frac{1}{1+r} Y_1 \qquad (48)$$

We could set up a similar Lagrangian function as in section A. Such a system has 13 unknowns: C_0, C_1, L_0^S, L_0^L, H_0, L_1^S, L_1^L, H_1, λ, Y_0, Y_1^S, Y_1^L, Y_1, in 13 equations:

a) FOC for C_0

b) FOC for C_1

c) FOC for L_0^S

d) FOC for L_0^L

e) FOC for L_1^S

f) FOC for L_1^L

g) intertemporal budget constraint (48)

h) production function (44)

i) production function (45)

j) production function (46)

k) total future output (47)

l) time constraint (42)

m) time constraint (43)

The fundamental goal of this analysis would be to find the determinants of the equilibrium rate of interest r. We can again consider a homogeneous or heterogeneous-agent model. For a homogenous model, only one more equation is required. Namely, that the individual saving is zero, i.e. $C_0 = Y_0$. As can be seen, present goods cannot be saved, but present labour can in the form of a longer production process. Although we will not solve this problem, several observations will surely emerge.

First, the equilibrium rate of interest will be determined by various parameters of the model. It will depend not only on the time preference parameter ρ, but also on the productivity parameters α, β, γ. A diminishing marginal product of capital (i.e. of longer methods) might appear here, if $\beta + \gamma < 1$. Thus, the entire picture of this economy might be represented by convex (consumption) indifference curves and a concave investment opportunity line, whose shape depends not only on the productivity of longer methods (46), but also on the diminishing marginal productivity of labour. Furthermore, its shape will be surely affected by the subjective discount rate. As a result, the natural rate of interest in this more comprehensive model will depend on the time preference and productivity phenomena.

To conclude this appendix, the inclusion of the intertemporal substitution of labour might open new fields in the analysis of the natural rate of interest. In section A, the natural rate of interest was determined by purely subjective phenomena, even though we generated a typical PPF curve, whose nature was, however, also purely subjective.

In section B, we allowed for a decreasing marginal productivity of labour. We suggested that this phenomenon must modify our analysis from the previous section. In final section C, the idea of higher productivity of roundabout methods was introduced. It is highly probable, that the natural rate of interest in such a model must be co-determined by the time preference and diminishing marginal productivity (of longer methods).

Supplement 1 to section A. Endowment point after a change in the subjective discount rate

In this supplement, we will prove that after the change in the subjective discount rate, the endowment point must move along the original budget line.

The optimum amount of present leisure time (equation 19) might be expressed as:

$$H_0^* = \frac{\dfrac{(1+r)W_0 + W_1}{(1+r)W_0}}{\dfrac{b(1+\rho)+1+\rho+b+1}{b(1+\rho)}} \tag{49}$$

$$H_0^* = \frac{(1+r)W_0 + W_1}{(1+r)W_0} \times \frac{b(1+\rho)}{(1+\rho)(b+1)+b+1} \tag{50}$$

$$H_0^* = \frac{(1+r)W_0 + W_1}{(1+r)W_0} \times \frac{b(1+\rho)}{(1+b)(2+\rho)} \tag{51}$$

Similarly for the optimal future leisure time, equation (21) might be written as:

$$H_1^* = \frac{W_0}{W_1}\frac{(1+r)}{(1+\rho)}\frac{(1+r)W_0 + W_1}{(1+r)W_0} \times \frac{b(1+\rho)}{(1+b)(2+\rho)} \tag{52}$$

$$H_1^* = \frac{(1+r)W_0 + W_1}{W_1} \times \frac{b}{(1+b)(2+\rho)} \tag{53}$$

Now, let us define the present value of the income stream. From the intertemporal budget constraint (Equation 2) and time constraints (3) and (4), it is clear that:

$$PV = W_0(1 - H_0^*) + \frac{W_1(1 - H_1^*)}{(1+r)} \tag{54}$$

As can be seen, we consider only optimal levels of leisure in both periods. Using (51) and (53):

$$PV = W_0 - \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b(1+\rho)}{(1+b)(2+\rho)} + \frac{W_1}{(1+r)} - \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b}{(1+b)(2+\rho)} \tag{55}$$

$$PV = W_0 + \frac{W_1}{(1+r)} - \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b}{(1+b)(2+\rho)}(2+\rho) \tag{56}$$

$$PV = W_0 + \frac{W_1}{(1+r)} - \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b}{(1+b)} \tag{57}$$

The present value expression from Appendix 2 is here adjusted for the last term. However, if we exclude leisure time from the utility function (i.e. b=0), it will be perfectly the same as in Appendix 2. Furthermore, PV of the (optimum) income stream does not depend on the subjective discount rate. This is the crucial result of the foregoing analysis:

$$\frac{\partial PV}{\partial \rho} = 0 \tag{58}$$

In the first round, a change in the subjective discount rate does not alter the real interest rate. If both the PV and the interest are constant, then the new endowment point must lie on the original budget line, because neither the slope of the budget line (r is constant), nor the position of the budget line (PV is constant) changed. In other words, the new budget line must coincide with the initial one, even though the endowment point is at a different position.

Alternatively, we can demonstrate that the endowment point ($Y_1 = W_1L_1^*$; $Y_0 = W_0L_0^*$) is moved in the same direction as is the slope of the budget constraint. This slope (dC_1/dC_0) is specifically given by –(1+r) as can be seen from the explicit form of the budget constraint:

$$C_1 = (1+r)\mathrm{W}_0\mathrm{L}_0 + \mathrm{W}_1\mathrm{L}_1 - (1+r)C_0 \tag{59}$$

We will solve this problem with the help of the optimum leisure time rather than labour supply. It will be also useful to define the following relationship:

$$W_0 H_0^* = W_0 - W_0 L_0^* \qquad (60)$$

The last term in (60) is the optimum present income. Furthermore, the response of present income to a change in the subjective discount rate is, using (60), given by:

$$\frac{\partial (W_0 H_0^*)}{\partial \rho} = -\frac{\partial (W_0 L_0^*)}{\partial \rho} \qquad (61)$$

Similar relationship holds for the future period:

$$\frac{\partial (W_1 H_1^*)}{\partial \rho} = -\frac{\partial (W_1 L_1^*)}{\partial \rho} \qquad (62)$$

Using (50), we can find the optimal response of the present leisure time to a change in the time preference (in sense two):

$$\frac{\partial (W_0 H_0^*)}{\partial \rho} = \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b}{(1+b)} \times \frac{(2+\rho)-(1+\rho)}{(2+\rho)^2} \qquad (63)$$

$$\frac{\partial (W_0 H_0^*)}{\partial \rho} = \frac{(1+r)W_0 + W_1}{(1+r)} \times \frac{b}{(1+b)} \times \frac{1}{(2+\rho)^2} \qquad (64)$$

Using (53), we can find the optimal response of future leisure time:

$$\frac{\partial (W_1 H_1^*)}{\partial \rho} = -\left[(1+r)W_0 + W_1\right] \times \frac{b}{(1+b)} \times \frac{1}{(2+\rho)^2} \qquad (65)$$

Applying (61) and (62) and dividing (65) by (64), the optimal relative change in future income ($Y_1 = W_1 L_1^*$) to present income ($Y_0 = W_0 L_0^*$), i.e. the movement of the endowment point A, in response to a change in the subjective discount rate is given by:

$$\frac{\dfrac{\partial (W_1 L_1^*)}{\partial \rho}}{\dfrac{\partial (W_0 L_0^*)}{\partial \rho}} = -(1+r) \qquad (66)$$

Thus, a change in the subjective discount rate will move the endowment point along the original budget line. Q.E.D.

<u>*Supplement 2 to section A. Endowment point after a change in the real interest rate*</u>

In this second supplement, we will demonstrate that the response of the endowment point to a change in the real interest rate is more complicated than the response to a change in ρ.

Using (51), a response of present leisure time to a change in the interest rate is given by:

$$\frac{\partial(W_0 H_0^*)}{\partial r} = \frac{b(1+\rho)}{(1+b)(2+\rho)} \times \frac{W_0(1+r) - \left[(1+r)W_0 + W_1\right]}{(1+r)^2} \quad (67)$$

$$\frac{\partial(W_0 H_0^*)}{\partial r} = \frac{b(1+\rho)}{(1+b)(2+\rho)} \times \frac{-W_1}{(1+r)^2} \quad (68)$$

Using (53), a response of future leisure time might be expressed as:

$$\frac{\partial(W_1 H_1^*)}{\partial r} = \frac{b}{(1+b)(2+\rho)} \times W_0 \quad (69)$$

Applying (61) and (62) and dividing (69) by (68), we can easily derive the relative movement of the endowment point after a change in the real interest rate:

$$\frac{\dfrac{\partial(W_1 L_1^*)}{\partial r}}{\dfrac{\partial(W_0 L_0^*)}{\partial r}} = -\frac{W_0}{W_1}\frac{(1+r)^2}{(1+\rho)} \quad (70)$$

If $W_0 = W_1$ and $r = \rho$, the derivative above is equal to (1+r) in absolute value. Hence, after the decrease (or increase) in the interest rate, the new endowment point would lie on the original budget line. If $r < \rho$, the derivative is lower than (1+r). In Figure no. 1_A3, we assumed that $r < \rho$ and $W_0 < W_1$ (or $W_0 = W_1$). Thus, the new endowment point must be below the original budget line because the directional shift of the endowment point has a lower slope than the budget line. If $r > \rho$, the derivative would be higher than (1+r) and the new endowment point would lie above the initial budget line.[27]

[2727] Condition (70) implies that the response of the endowment point to a change in the real interest rate depends on the position of the original endowment point. If the initial endowment point is above the 45º line, the new endowment point will be below the original budget line and vice versa. The reason lies in the fact that point A is above the 45º line (i.e. $Y_1 = W_1 L_1^* > Y_0 = W_0 L_0^*$), if $W_1 > W_0$ and $r < \rho$ (and hence $L_1^* > L_0^*$) or if $W_1 > W_0$, $r > \rho$ (!) and the growth rate of wages (W_1/W_0 - 1) exceeds the difference between r and ρ.