The Austrian Theory of the Natural Rate of Interest: A Neoclassical Critique, Part Four

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5. DYNAMICS AND THE INFINITE HORIZON

In this part, we will relax the assumption of only two-period live. First, we will assume that the representative individual lives for T-periods. Next, we let T go to infinity. And finally, we will explore the behaviour of an economy in continuous time rather than in discrete time. The extensions made in this section will shed some light on the problems of the interest theory that are obscured in a two-period model or that cannot emerge there at all. On the other hand, the ideas developed here are much more difficult or sometimes virtually impossible to plot in a graph.

We will start with an optimal allocation of consumption of a representative individual who lives for T periods. Recall the lifetime utility function represented by equation (1). To find the optimum consumption path of an individual, we have to add his intertemporal budget constraint (IBC). If his lifetime is T, a usual form of the IBC might be represented by (Olson, Bailey 1981:9):1

$$C_0 + \frac{1}{1+r}C_1 + \frac{1}{(1+r)^2}C_2 + \frac{1}{(1+r)^3}C_3 + ... + \frac{1}{(1+r)^T}C_T =$$

$$= Y_0 + \frac{1}{1+r}Y_1 + \frac{1}{(1+r)^2}Y_2 + \frac{1}{(1+r)^3}Y_3 + ... + \frac{1}{(1+r)^T}Y_T \quad (28)\ \mathbf{(29)}$$

Yt denotes real income at time t. Real interest rate r is assumed to be constant over time. For varying interest rate across time, (29) is modified to (30).

$$\mathrm{C}_0 + \frac{\mathrm{C}_1}{(1+\mathrm{r}_1)} + \frac{\mathrm{C}_2}{(1+\mathrm{r}_1)(1+\mathrm{r}_2)} + ... + \frac{\mathrm{C}_T}{(1+\mathrm{r}_1)(1+\mathrm{r}_2)...(1+\mathrm{r}_\mathrm{T})} =$$

$$= \mathrm{Y}_0 + \frac{\mathrm{Y}_1}{(1+\mathrm{r}_1)} + \frac{\mathrm{Y}_2}{(1+\mathrm{r}_1)(1+\mathrm{r}_2)} + ... + \frac{\mathrm{Y}_T}{(1+\mathrm{r}_1)(1+\mathrm{r}_2)...(1+\mathrm{r}_\mathrm{T})} \quad (29)\ \mathbf{(30)}$$

Setting up a simple Lagrangian for (1) and (29), one can show (see Appendix 4) that FOCs of this problem lead to the following Euler equation for any time t and t+1:

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{1+\rho}{1+r} \qquad (30)\ \mathbf{(31)}$$

Equation (31) implicitly defines the optimum path of consumption of a representative individual and it should be familiar from the two-period model.2 Alternatively, the solution of this optimization problem can be expressed as:

[1] The derivation of this particular intertemporal budget constraint is presented in Appendix 5.
[2] If we used equation (30) instead of (29), the Euler equation would be $u'(C_{t+1})/u'(C_t) = (1+\rho)/(1+r_{t+1})$

$$\frac{u'(C_t)}{u'(C_0)} = \left(\frac{1+\rho}{1+r}\right)^t \qquad (31)\ \mathbf{(32)}$$

Equation (32) allows us to reconsider the Misesian statement that for zero time preference (in sense two, i.e. ρ=0) and positive interest rate, acting man will be postponing his consumption to an indefinite future forever. If we extend the time horizon to infinity and set ρ=0, we get (Olson,Bailey 1981:12):

$$\lim_{T\to\infty} \frac{u'(C_T)}{u'(C_0)} = \lim_{T\to\infty} \frac{1}{(1+r)^T} = 0 \qquad (32)\ \mathbf{(33)}$$

Equation (33) suggests that for a positive rate of interest and zero subjective discount rate, in optimum the ratio of the marginal utility from consumption in infinity and from consumption today is zero. This can be either achieved by an infinite marginal utility in the present or zero marginal utility in infinity. If the utility function satisfies usual Inada conditions, infinite marginal utility is achieved by zero consumption and conversely zero marginal utility is obtained by infinite consumption. Alternatively, if we allow for a subsistence level Cmin, the infinite MU is reached at this level. Correspondingly, satiation level Cmax would lead to zero MU.

Equation (33) therefore requires that in optimum, the present consumption must be depressed to a negligible level provided that the budget constraint (29) (or better budget constraint A5_13 in Appendix 5) does not allow for an infinite consumption in infinity (Olson, Bailey 1981:12). As a result, all income should be postponed to infinity because the compounding of interest in infinite horizon may expand consumption beyond all limits. This outcome is so attractive that every unit of present consumption should be postponed to this remote future.3 Hence, it seems that the Misesian analysis should hold in the infinite horizon model, since positive rate of interest is inconsistent with zero time preference (in sense two). The two-period model has therefore hidden this important outcome and our critique of Mises was inaccurate.

Moreover, positive interest rate and zero subjective discount rate cannot create long run equilibrium in the production part of the economy if capital exhibits diminishing returns. The never-ending postponing of consumption implies that people save (almost?) entire income. Huge saving and immense accumulation will extend the capital stock beyond all limits. This process should eventually depress the marginal product of capital to zero along with the interest rate. In the end, the real interest rate and the subjective discount rate must coincide at the zero level and the accumulation of capital stops.

[3] In Appendix 4, we will demonstrate that present consumption need not be curtailed to negligible levels, even if the subjective discount rate is zero and the interest rate is positive. The reason lies in the fact that infinite consumption in infinite horizon might be easily achieved just with moderate present savings. As a result, equation (33) is satisfied due to zero numerator rather than infinite denominator. This Appendix will also reveal further problems of the approach presented in Olson, Bailey (1981). The key point is that they underestimated interesting and unexpected implications, if the time horizon is extended to infinity. However, our correction of their theory will not support Misesian PTPT. Rather the opposite is true. It will be shown that positive interest rate accompanied by zero time preference might lead to non-zero present consumption even under weaker assumptions than Olson and Bailey believed. Yet, these corrections are not presented in the main text because here we are mainly interested in problems of the Misesian PTPT. The inclusion of long proofs will make our exposition less lucid. However, even the first approximation of Bailey and Olson suffices to reveal fundamental problems in PTPT, although their theory is not perfectly accurate.

However, in real world we usually observe positive real interest rate. At the same time, we do not witness a radical curtailment of present consumption. For Olson and Bailey (1981) this is an explicit evidence for the existence of positive time preference (ρ>0). As a result, their approach leads to similar conclusions as made by L. von Mises. In equilibrium, the interest rate must be equal to the time preference otherwise all consumption will be postponed to an indefinite future.[4]

The foregoing analysis will become even more obvious, if we introduce a particular form of the utility function. Consider e.g. the CRRA form. According to (19), equation (31) can be represented as:

$$\frac{C_t}{C_{t+1}} = \left(\frac{1+\rho}{1+r}\right)^{1/\theta} \qquad \textbf{(34)}$$

and equation (32) as

$$\frac{C_0}{C_t} = \left(\frac{1+\rho}{1+r}\right)^{t/\theta} \qquad (33)\ \textbf{(35)}$$

If we expand the time horizon to infinity and set ρ = 0, (35) yields:

$$\lim_{T\to\infty}\frac{C_0}{C_T} = \lim_{T\to\infty}\frac{1}{(1+r)^{T/\theta}} = 0 \qquad (34)\ \textbf{(36)}$$

If the time preference was zero and the real interest rate was positive, our individual would be in optimum either with zero present consumption or with infinite consumption in infinite future. As a result, it seems that the only stable outcome of this analysis is that the real interest rate is perfectly equal to the subjective discount rate (r = ρ) and the optimum consumption stream is smoothed over time ($C_0 = C_1 = \ldots C_T = \ldots$).

However, in actual world we observe an increasing shape of the consumption profile. In other words, consumption is not constant as it grows across time at some definite rate that seems to be quite stable over very long periods of time. It can be shown that in standard growth models with labour-augmenting technological progress, consumption (per worker) may grow at the same rate as income (per worker). And this rate is equal to the rate of technological progress g.[5]

As a result, our Euler equation (35 or 36) can be consistent (owing to the technological progress) with infinite consumption in infinity, even if present consumption is not depressed to a negligible level. Suppose that the growth rate of consumption (and the technological progress) is g = 2 %. For a logarithmic utility function (θ=1), this implies that the difference between the real interest rate and the subjective discount rate is roughly 2 % as well.

It can be shown (using 34) that the optimum growth rate of consumption must approximately obey the following expression:

$$\frac{\Delta C_{t+1}}{C_t} = \frac{r-\rho}{\theta} \qquad (37)$$

[4] ??? offered a different explanation for the Euler equation (33) to hold. Consider again the left hand part of equation (33). If tastes of people deteriorate over time, marginal utility in very remote future may converge to zero. Thus, the Euler equation (33) might be satisfied even with finite consumption in the future.
[5] Acemoglu… Growth Theory

Thus, if consumption grows at the rate of technological progress g, and if this rate is lower than the real interest rate r (which is required for a dynamically efficient economy),[6] equation (37) (or 34) might be satisfied even for positive real interest rate and zero time preference (ρ = 0), if θ is big enough. This is what Olson and Bailey called a "drastically diminishing marginal utility"(1981:19). We already know that high θ is equivalent to low elasticity of substitution and highly curved utility function. E.g. if g=2 %, r = 4 % and ρ=0 %, the only θ consistent with equation (37) is equal to 2.[7]

Thus, we demonstrated that the positive interest rate is consistent with zero time preference (in sense two) even in an infinite time horizon. However, the model requires an exogenous growth in income endowment and sufficiently convex indifference curves (high θ). Misesian economists would probably argue that an increasing labour income endowment violates the key assumption of the ERE. We may reply again that such a shape of the income stream is dominant in modern economies, so our model might accurately represent actual world.[8]

Misesian argument about the equality of the interest rate and time preference (ρ) does not hold, if the time shape of income is increasing. To display this situation graphically, consider a time profile of income in the two-period model (Figure no. 30, panel a) which is replicated every period. In the infinite horizon model too, (MRS-1) and rE might be positive even for zero time preference in sense two. Even if people do not prefer the given satisfaction to be delivered as soon as possible, the real interest rate might be positive and all consumption will not be postponed to infinite future.

We may add that the economic logic for high θ and a positive difference between r and ρ and r and g runs as follows. If the income increases over time and people have high θ, their preferred profile of consumption is rather smoothed. As a result, they prefer their high future income to be moved closer to present. They do not save very much, which increases the real interest rate both above the subjective discount rate and above g.

This model can be also used to give credit to Mises´s critique of J. Schumpeter. Mises believed that people a-priori prefer the given satisfaction to be achieved as soon as possible. He therefore claimed that there must be always time preference even in a stationary economy (ERE in his system) and hence a positive rate of interest. If government or the banking system artificially depresses the real interest rate to zero, a gradual consumption of capital should emerge. Our model gives similar predictions. According to (37), zero real interest rate and positive subjective discount rate favours present consumption over future consumption. As a result, the optimum profile of consumption is decreasing. People consume today at the expense of future and the capital stock must gradually fall. The artificially depressed interest rate should progressively exhaust the entire capital stock in the economy. The corresponding optimum profile of consumption is displayed in Figure no. 36. In this particular respect, Mises was perfectly right.[9]

In the two-period model, we demonstrated that the natural rate of interest might be negative, either for initial endowment that deteriorates over time (Figure no. 18), or for a sufficiently decreasing flow of income (Figure no. 30b). By similar argument, we have shown that the natural rate of interest could be zero. However, Fetter(???), Mises(???) and Rothbard(???) used the infinite horizon approach to deny the possibility of zero (or even negative) rate of interest.

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[6] According to Bailey and Olson (1981:19), this seems to be an empirical fact.
[7] A thorough analysis of the growth in income and of the requirement on θ is presented in Appendix 4.
[8] In Appendix 4, we will show that infinite consumption in infinity, positive interest rate, zero time preference and positive (i.e. non-zero!) present consumption are attainable even if the flow of income is constant.
[9] Hayek (192???) dealt with consumption of capital in a greater detail…

Consider a piece of land that provides an infinite flow of services. Its present price is in equilibrium equal to the discounted sum of the flow of these services. To keep the analysis in real terms, if the given piece of land provides an eternal real income of 100 apples every year, its market price should be:

$$\mathrm{PV} = 100 + \frac{100}{1+r} + \frac{100}{(1+r)^2} + \ldots = 100\frac{1}{1-\frac{1}{1+r}} = 100\frac{1+r}{r} \qquad \textbf{(38)}$$

where r is the real rate of interest prevailing in the economy. Fetter and other PTPT authors claimed that for zero rate of interest, the market price of this piece of land (and any other perpetuity) would be infinite. This approach therefore provides an indirect proof that the interest rate can never (permanently) fall to zero (or even below zero).[10]

However, let us now demonstrate that there exists an important gap in their reasoning. The key problem is that they separated an analysis of a flow of income and the equilibrium level of the rate of interest. I. Fisher (1930) stressed many times that it is the particular flow of income that is of crucial importance in the interest theory. Here we will show that he was perfectly right.

As we have seen, the negative (or zero) rate of interest might be generated only for a decreasing flow of income, provided that the subjective discount rate is positive. Hence, the flow of perpetual income of 100 apples forever can never generate a zero or negative rate of interest. In this particular respect, Fetter was right.

However, suppose that the given piece of land provides a perpetual flow of income that falls at some definite rate g (e.g. g = -4 %). This means that the present output of apples is 100, the next year output of apples is 96, etc. The present price of land is then calculated as:

(37)

$$\mathrm{PV} = 100 + \frac{100(1+g)}{1+r} + \frac{100(1+g)^2}{(1+r)^2} + \ldots = 100\frac{1}{1-\frac{1+g}{1+r}} = 100\frac{1+r}{r-g}$$

$$\mathrm{PV} = 100 + \frac{100}{1+r} + \frac{100}{(1+r)^2} + \ldots = 100\frac{1}{1-\frac{1}{1+r}} = 100\frac{1+r}{r}$$

The sum of this infinite series converges, if the interest rate exceeds g.[11] Assume that the interest rate is zero, (39) then yields that the price of land is 100/0.04=2500. To make these calculations consistent with the Euler equation of (37), consider ρ=4% and θ=1 (because -4 % = (0 - 4 %)/1). From (39), a definite price of land is guaranteed, if g<r. And according to the Euler equation, r = ρ + θ*g. From (39) and the Euler equation, we can say that a very great number of combinations may generate situations that are totally unthinkable in Fetter´s theory of interest.

It can be explicitly said that Fetter´s argument is not valid. Even for positive time preference (ρ>0), the interest rate might be zero (or even negative) and the value of perpetual land will

[10] Obviously, equation (37) is a final step in deriving the sum of an infinite series, provided that the real rate of interest is the same in all periods. However, if the interest date differs over time, the sum need not be infinite, if the interest is zero in some periods but positive in others. However, its positive value is necessary in large enough periods to guarantee convergence of the total sum.

[11] Hence, a finite price of land requires that the economy is dynamically efficient. If the growth rate of income from land exceeded the rate of interest, the price of land would grow beyond all limits. Furthermore, zero interest rate is possible, but it must exceed the growth rate of income (in this case g must be negative).

not expand beyond all limits. The primary phenomenon is the flow of income. It influences not only the natural rate of interest, but also the definite price of the given asset generating this particular flow of income. In our example, a negative time shape of the income stream was perfectly consistent with zero (or even negative) rate of interest and a finite value of the given piece of land. Our simple example explicitly confirmed the Fisher´s statement that the analysis of the income stream can never be separated from the theory of the rate of interest otherwise we obtain incomplete and erroneous results like Fetter, Mises and Rothbard.

5.1 DYNAMICS IN CONTINUOUS TIME AND THE NATURAL RATE OF INTEREST

In the last section of this paper, we will generalize our findings obtained so far by introducing a continuous time model. The analysis of either finite or infinite horizon is most elegant and rigorous in the continuous time approach. We will see that many ideas of Böhm-Bawerk are reflected in modern dynamic analysis. Moreover, the dynamic analysis with continuous time will demonstrate again that the pure time preference theory is at least incomplete.

A representative consumer facing dynamic intertemporal decisions can be described by the following life-time utility function:

$$\int_0^T e^{-\rho t} u(C(t))dt \qquad (38) \quad \mathbf{(40)}$$

Equation (40) is a continuous-time version of our discrete-time model. It satisfies all the usual neoclassical (and Austrian) assumptions. People prefer present satisfaction to future satisfaction, hence ρ>0. Marginal utility is always positive and declines with higher consumption, therefore u´(C)>0 and u´´(C)<0 for all C. Positive first derivative for all levels of consumption also guarantees that more is always preferred to less.

This simple model allows us to extend the analysis of many topics mentioned in previous parts. Let us start with the Fisherian sailors shipwrecked with a definite amount of hard-tacks. A simple dynamic analysis can easily demonstrate that Fisher´s predictions about the optimal allocation of hard-tacks over time are imprecise (1930:???).

As we have already stressed, Fisher concluded that the interest rate in the hard-tack economy must be necessarily zero. However, he offered the following figures demonstrating possible consumption paths (Figure no. 37).

Let us now demonstrate that none of these are optimal. First, set up a typical dynamic optimization problem:

$$(39)\ \max U = \int_0^T e^{-\rho t} u(C(t))dt \qquad \text{s.t. } \int_0^T C(t)dt \le K \qquad \mathbf{(41)}$$

A representative sailor lives for T periods. His initial endowment of hard-tacks is K. It cannot be extended either by labour effort, by exogenous transfer payments or by productive investment. So the only problem is the optimal allocation of the initial endowment over time. Fisher explicitly demonstrated that the interest rate in this economy must be zero. The solution of this problem is given in Appendix 6 along with several technical comments. At this place, we just report the final solution for the CRRA utility function:

$$C(t) = \frac{\rho}{\theta} \frac{e^{\rho(\mathrm{T-t})/\theta}}{e^{(\rho/\theta)\mathrm{T}} - 1} K \qquad (40) \quad \mathbf{(42)}$$

Figure no. 38 depicts two optimal paths for different T. As can be seen, longer life requires lower consumption in every period. Furthermore, Figure no. 39 displays that more patient sailors (lower ρ) have a flatter profile of the optimum consumption. And finally, Figure no. 40 demonstrates that for given ρ, lower elasticity of substitution (higher θ) results in a smoother optimum consumption path.[12] None of these figures resembles the original pictures offered by Fisher. However, he could not have used the benefit of modern modelling techniques.[13]

As can be seen, constant consumption of hard-tacks over time in not optimal. The reason is the existence of positive subjective discount rate. Its presence requires that the un-discounted future marginal utility must be higher than present marginal utility. And this can be achieved only with lower consumption in the future (see Appendix 6 for technical details). Thus, the preference for present satisfaction over future satisfaction leads to a downward sloping profile of the optimum consumption path. However, even if the interest rate is zero, all hard-tacks are not consumed in the present. The reason lies in the diminishing marginal utility of consumption. This law requires that levels of consumption in two consecutive periods (whose distance is infinitely small in the continuous model) are very close to each other. Hence, we do not observe any dramatic jumps in consumption levels, but a smooth (though decreasing) path.

As a result, the existence of positive time preference and zero rate of interest do not result in a complete exhaustion of resources in the present as Mises would argue. The necessary break is performed by the law of diminishing marginal utility. This law is in turn derived from the idea that the given good is able to satisfy only wants of lower and lower intensity as its consumption rises in the given period. Hence, it cannot be optimal to move all goods to one period (be it present or future) and leave the needs in the rest of the life unsatisfied.[14] As can be seen, we derived a similar conclusion in a continuous-time model for lifetime T as we observed in a simple two-period model.

A more complicated dynamics would be achieved, if there was some minimum subsistence level of hard-tacks required every period till time T. Figure no. 41a displays the optimum path of consumption for this possibility. An extreme version and the most unfortunate one would be, if the initial stock was not big enough to preserve life till time T (panel b). However, in this case there are not much economic decisions to analyse.

The previous example was based on constant marginal productivity of capital, being zero in case of hard-tacks. Now, we extend our analysis from section 4, where we considered investment opportunity curve and diminishing marginal productivity of capital. The more advanced model of this section allows us to make several extensions. First, the continuous time version narrows the distance between two periods to an infinitely small lapse of time. Second, there is an infinite number of periods rather than only two. However, the Austrian idea that the time extension of the production process provides higher output is not present here either. So the concept of decreasing marginal productivity of capital will again take place

[12] As can be seen, ρ and θ have opposing effects on the optimum consumption path. Higher θ leads to a smoother profile of consumption, which, however, implies that present consumption is lower. Thus, for a fixed present endowment, higher θ has similar effects as lower time preference. This could be compared with the discussion in Appendix 4. Here, it was derived that if the income stream is increasing over time, higher θ results in higher present consumption, as the individual is trying to move higher future income closer to the present. Thus, higher θ had similar effects as higher time preference. As can been seen, assuming fixed present endowment high preference for consumption smoothing (high θ) motivates the consumer to spread his consumption over the entire planning horizon. Yet, this leads to lower present consumption.

[13] A similar statement can be found in Woodford (2003:5) about Wicksell and Hayek when considering their contribution to (modern) monetary theory.

[14] Similar objection along with the presentation of his own position can be found in Knight (On Mises ???:???)

only in capital´s breadth, not in its height. Nevertheless, many interesting insights might be found in this model as well.

An infinite continuous time version of the simple Fisherian model from section 4 closely resembles the Ramsey-Cass-Koopmans model.[15] It is developed in Appendix 7. Its building blocks consist of a representative consumer/dynasty maximizing lifetime utility,[16] the law of motion of capital and the production function with usual properties, especially with the diminishing marginal productivity of capital. Its solution is derived in Appendix 7. Here, we just use the fact that for a positive labour-augmenting technological progress growing at the rate of g, the steady state general equilibrium requires:[17]

$$MPK-\delta=\rho + \theta g \qquad (40) \qquad \mathbf{(43)}$$

The interest rate in the economy must be so adjusted that its steady state value r* guarantees condition (43), thus $MPK-\delta = r^* = \rho + \theta g$. Surprisingly, all three terms in (43) might be associated with one of the three causes of interest in the Böhm-Bawerkian theoretical framework. Positive and diminishing MPK denotes the idea that roundabout (i.e. capital using) methods of production give higher output, but at a decreasing rate. This term indicates the productivity element; it is the third cause of interest. The subjective discount rate ρ, as was discussed many times before, stands for the undervaluation of future wants; it represents the second cause of interest. And finally, the first cause is hidden in the term θg, even though this might not be obvious at the first sight. In the steady state (on the balanced growth path) of this model, the income per person grows at the rate of technological progress g. So the income endowment of each individual grows at this particular rate. Every individual is wealthier every subsequent period. This gives present goods additional premium over future goods in the minds of people, because future is better provided for than present. As we know, Böhm-Bawerk identified this phenomenon as the first cause for interest.

Equation (43), expressed as $MPK-\delta = r^* = \rho + \theta g$, might be used to demonstrate again that zero rate of interest is possible even for a positive subjective discount rate and conversely, positive rate of interest can be generated even for a zero subjective discount rate. The first possibility may emerge, if the income endowment falls at a sufficiently rapid pace. Consider zero population growth (ZPG), logarithmic utility function ($\theta=1$), $\rho = 4\%$ and $g = -4\%$. This set of parameters leads to $r^* = 0$ %. Notice that this combination is virtually the same as for the discrete time model.[18] However, here we formally closed the model by adding the productivity element.

Thus, zero natural rate of interest is achieved, if $\rho = -\theta g$. If the population growth is zero and the condition for the convergence of lifetime utility is satisfied, $\rho-n-(1-\theta)g>0$ (see Appendix 7, condition A7_14), zero interest rate requires that the technological progress (and the growth rate of income) is negative. In other words, even if people prefer the given satisfaction to be achieved as soon as possible ($\rho>0$), the natural rate of interest might fall to zero provided that they are becoming poorer over time due to the exogenous decay in their income endowment. Owing to this decline in income over time, wants of higher intensity are expected to remain unsatisfied in the future. As a result, present goods might lose their superiority over future

[15] Ramsey (1927???), Cass (1963???), Koopmans (1963???) comments

[16] Here, we assume CRRA instantaneous utility function. Convergence of life-time utility then requires that $\rho -n-(1-\theta)g>0$

[17] δ is the depreciation rate. Alternatively, we can write $MPK - \delta = r^* = \rho + \theta g$

[18] Even a negative natural rate of interest can be achieved with positive ρ. Set e.g. $\rho = 4$ %, $\theta =1$, $g = - 5\%$.

goods and the real natural rate of interest could be easily depressed to zero as people are rushing to move their present income to the less abundant future via huge saving.[19]

It can be seen again, that the time shape of income is crucial for the eventual level of the natural rate of interest. Even this extended general equilibrium model gives the result repeated many times before. At the same time, an incredibly deep insight in the works of early economists must be praised. Böhm-Bawerkian three causes of interest are still present in the modern economic growth models, although under the disguise of different names. Fisher´s primacy of the flow of income is present here as well. It is absolutely fascinating that their intuition arrived at similar results as modern sophisticated models, whose solution requires many complicated steps and the employment of dynamic optimization techniques.

It should be stressed that the RCK model is also consistent with positive rate of interest and zero subjective discount rate.[20] Consider again equation (43) and the condition for the convergence of lifetime utility A7_14. For ZPG and $\rho=0$ the following condition must be satisfied: $\theta>1$ and $g>0$. As was discussed in the discrete-time version, positive growth in income endowment is required along with a relatively low elasticity of substitution. Then, people will try to move their higher future income to the present via reduced saving, if they prefer a smoothed path of consumption ($\theta>1$). Thus, the interest rate may increase to the positive region, even if the subjective discount rate is zero. A relative abundance of goods in the future resulting from positive technological progress gives the present goods premium over future goods, even though there is no underestimation of future wants.[21] As a result, if the second Böhm-Bawerkian cause for interest is not present ($\rho = 0$), the first cause must be operative to obtain a positive rate of interest (together with positive marginal productivity of capital).

As we can see again, the PTPT is a special case of a more general theory. Natural rate of interest at the steady state would be solely determined by the time preference (in sense ii), if the income per person was stable ($g = 0$ %). This situation closely resembles Misesian ERE. However, if income varies over time, the subjective discount of future utilities is not the only determinant of the natural rate of interest.

Let us now analyse in a more detail the behaviour of the natural rate of interest in an economy with non-constant income. Hayek (PTC) in his famous book envisioned an idea of dynamic equilibrium, i.e. the equilibrium for a growing economy. In this particular respect, modern growth models are much closer to the Hayekian vision compared with the Misesian theory. Furthermore, the idea of dynamic equilibrium seems to be much closer to real world economies, since in normal times they are growing. Thus, the assumption of the growing income endowment is of particular importance in the theory of interest. The PTPT authors neglected this very important aspect, which resulted in the fact that their approach seems to be rather incomplete.

The behaviour of the natural rate of interest in an economy with non-constant income is best understood, if the economy is not at its steady state (or balanced growth path-BGP). Yet, we will also consider an economy that is initially at the BGP but suddenly it is hit by time preference or technological shock. We will discuss again the relative importance of time

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[19] A path of the economy to the zero natural rate of interest is displayed in Appendix 7, section G, Figure no. 28_A7. Furthermore, Section G in Appendix 7 discusses other interesting phenomena in this case of contracting economy.

[20] At first sight, it seems that the integral in (40) should diverge for $\rho=0$. However, for CRRA with $\theta>1$ and increasing consumption this will not happen as is shown further in the text and in Appendix 5B, **Figure no. 1_A5 and 2_A5**.)

[21] This fact is stressed by Bloome (???) in justifying the discounting of wants of future generations.

preference and productivity in determining the natural rate of interest. Before using the RCK model, however, we will introduce a very insightful approach developed by Hayek (PTC).

Hayek (1941) in his very difficult book spent many pages to investigate the relative importance of time preference and productivity in an economy that is accumulating capital. He used an ingenious extension of the Fisher model (**see Figure no. 42 (41)**). As we can see, his model contains indifference curves and investment opportunity lines. However, he tried to make the Fisher model more dynamic so he added a 45º line representing the same amount of consumption in both periods. The axes represent any two consecutive periods (t and t+1).[22] Furthermore, his curves are more consistent with net concepts rather than gross concepts. Thus, e.g. a movement along the opportunity line portrays a net increase in future return.

Hayek (PTC) assumed that for shorter periods, the investment opportunity line is less curved than the indifference curve. The reason lies in the fact that investment made within a relatively short period of time cannot much affect the schedule of the marginal productivity of capital as the investment is only a negligible part of the entire capital stock. On the other hand, the given amount of saving in the short period represents a relatively significant part of income in that period so "the sacrifice of successive parts of the income of this interval of time in the interests of the future will meet with a rapidly increasing resistance"(1941:233).[23]

Point P represents an invariant flow of income earned only due to permanent resources (i.e. land and labour) without any use of capital. At this start, capital as a factor of production is very productive, so the investment opportunity curve has a large slope. In the first period, a sacrifice of present consumption may generate a very high increase in future output. For a reasonable rate of time preference (in sense ii), the economy might move to a point like A. Due to a relatively low curvature of the investment opportunity curve, the interest rate is determined by the productivity of capital rather than by the time preference (in sense i). Time preference (in the sense of MRS-1) will only adjust to the given rate of return.

To determine the position of the economy in the next period let us shift the system one period forward (a movement from point A to point B). Notice that as future becomes present, consumption is higher compared with the situation in which no permanent resources were used in the creation of capital (B versus P). Hayek assumed that the marginal productivity of capital gradually falls, so the next period investment curve is not as favourable as the previous one. As we can see, the economy finds the new equilibrium at point C, closer to the 45º line.

This process might be repeated many times. Along the path to the steady state equilibrium, the interest rate is determined by the productivity of capital. According to Hayek, time preference will only affect the rate of saving and the speed at which the capital will be accumulated (PTC:???). However, as the marginal product of capital gradually falls, the next period increments in output are still lower and lower. Finally, the process stops at point E. At this point, there is no net accumulation of capital and consumption is stable over time. Furthermore, at this point and only at this point of long-run equilibrium, the natural rate of interest is determined by the time preference (the slope of the indifference curve at the 45º line). It is also obvious that for more patient people, point E is posited further from the origin and the process of capital accumulation lasts for a longer period of time.

As we can see, Hayek introduced a very interesting and novel theory. Over the process of the accumulation of capital, the natural rate of interest depends on the productivity of capital. However, at the eventual steady state, it is determined solely by the time preference. What is

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[22] Hayek used horizontal axis for period t+1 and vertical axis for period t. We swapped the axes to make his model more comparable to our approach in this paper.

[23] In this particular respect, Hayek accepted the theory of Frank Knight. See e.g. Knight (???, On Mises …)

even more interesting, modern RCK model gives analogous predictions. If we look at the convergence of the economy in this model, the story seems to be very similar.

Figure no. 43 portrays a convergence process in the RCK model. For simplicity, we assume zero technological progress, g=0. The economy starts with capital stock k(0) and consumption c(0). It could have moved here after a sudden and unexpected decrease in the subjective discount rate.[24] The shape of the saddle path along which the economy moves to the steady state and the speed of convergence depend on θ and ρ - the time preference and elasticity of substitution parameters, where the first parameter affects the slope of the indifference curve at the 45º line and the second parameter the curvature of the indifference curve at any point (apart from the 45º line). Similar predictions were made by Hayek. In the end, the economy should reach the steady state level of the natural rate of interest $r^*=\rho$. Hence, in the steady state the natural rate of interest depends solely on the time preference (in sense ii). However, along the convergence path, the interest rate behaves according to the diminishing marginal product of capital (Figure no. 43b). As we can see, this behaviour is also in line with the Hayek´s model. As regards the evolution of consumption, it gradually increases as the economy accumulates more capital. In the eventual steady state, accumulation stops and consumption is stable. It is quite fascinating that there is virtually no difference in predictions of the intuitive approach of Hayek and this modern growth model.[25]

From equation (43), it is perfectly clear that the rate of technological progress positively affects the steady state level of the natural rate of interest. Let us now consider a sudden increase in the growth rate of technological progress from g_1 to g_2. It is obvious that the steady state level of the natural rate of interest rises. However, what is the evolution of this variable in the transition process? Figure no. 44 shows that the natural rate of interest will gradually increase to the new steady state level. As can be seen in panel b, in the transition process, the demand for capital grows faster than the supply of capital. The key reason is that the marginal product of capital grows immediately at the faster rate of g_2, whereas the supply of capital is driven up by a higher growth rate in income, which only gradually rises to the new steady state (BGP) level of g_2. In other words, the growth rate in income and hence saving and the resulting growth rate in the supply of capital only gradually increase from g_1 to g_2.[26]

Furthermore, it should be stressed that every intersection of the demand and supply in Figure no. 43 represents an equilibrium natural rate of interest $r_{E,t}$ in that given period. The steady state level r^* is only one special equilibrium which is characterised by the fact that it is invariant over time. One may say that $r_{E,t}$ is a static equilibrium in the given period, whereas r^* is the very long-run dynamic equilibrium. However, any artificial attempt to narrow the difference between $r_{E,t}$ and r^* at any moment in the transition period should result only in the disparity between demand and supply of capital and consequently in the misallocation of resources. More on this will be said in chapter 4, which deals with business cycle considerations in a growing economy.

At the new steady state level, the natural rate of interest is higher $r_2^* = \rho + \theta g_2$. Faster technological progress affects the new level of the natural rate of interest via two channels. The first one is higher (growth rate of) productivity of capital. This can be identified with the third Böhm-Bawerkian cause for interest. However, the more fundamental is the channel of

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[24] In Appendix 7, this model is derived and the whole discussion about the fall in ρ is provided (section D).

[25] However, there is one exception. MPK at the given time depends on the level of capital, which in turn depends on the speed of convergence λ. However the size of λ is also influenced by ρ. Thus, it might be said that the natural rate of interest along the convergence path is affected not only by the marginal productivity of capital but, to some extent, also by the subjective discount rate.

[26] In Appendix 7, this model is derived and the whole discussion about the increase in g is provided (section D).

the first cause of interest. Higher growth rate of income-endowment (g_2) gives present goods higher premium over future goods owing to the abundance of future goods in the more remote periods. If the rate of technological progress suddenly rises, the determining factor in the transition process is the marginal product of capital. In the eventual steady state level, it is the time preference (in sense one, i.e. the joint influence of the first and the second cause of interest) that is of key importance. Nonetheless, as was remarked by Brown (???) increasing income-endowment associated with the first reason for interest is caused by growing productivity, hence it is the cooperation of the time preference and productivity that ultimately determine the given natural rate of interest.

In other words, without increasing productivity the first cause will not be operative at the steady state of r*. Yet, it is quite difficult to attach this phenomenon to the third cause of interest as it mainly reflects higher (but diminishing) (marginal) productivity of longer methods of production. Thus, we may say that the role of productivity at the eventual steady state is performed through the first cause of interest, i.e. through a permanently increasing income endowment.

Final question, studied even by Fisher (1930:???) in a great detail, is the evolution of the natural rate of interest, if the increase in technologies takes the form of a one-time shock. In other words, instead of the growth rate, we will assume a sudden one-time increase in the level of technologies.[27] The prediction of the RCK model is given in Figure no. 44. Higher level of technologies will immediately increase the marginal product of capital and consequently the natural rate of interest. However, the gradual accumulation of capital resulting from higher income should gradually decrease the natural rate of interest to the initial level (Figure no. 45). At this state, it is again determined solely by the subjective discount rate (i.e. time preference in sense ii), not by the marginal productivity of capital as in the transition process.[28]

As we have seen in section ???, Fisher suggested that higher average income might be associated with lower impatience. Thus, he speculated that higher level of technologies should eventually depress the natural rate of interest below its initial level due to improved living standards. This possibility is depicted by a dashed line in Figure no. 45. Higher income eventually decreases the subjective discount rate, which is the ultimate determinant of the natural rate of interest at the steady state.[29] Thus, the dynamics of the natural rate of interest after a sudden increase in the level of technologies might be very complicated.

In the previous section, we noticed that PTPT authors (Mises, Rothbard) deny that the productivity shock should have any influence on the rate of interest. Here we demonstrated that the answer depends on the nature and permanence of the shock. If the productivity shock is a discrete positive jump, the natural rate of interest suddenly increases. It then gradually falls back to the level dictated by the time preference $r^*=\rho$. This decline, however, is most

[27] For simplicity, we assume g=0%. The general discussion about the outcomes of this model is provided in Appendix 7. In this appendix, we also discuss the most important differences between the increase in g and in A (section D). Some of them (e.g. impact on the saving rate and the role of θ) are very important. However, to keep a continuous flow of our discussion about the natural rate of interest undisturbed, they are all postponed to Appendix 7.

[28] In the transition process, the representative consumer is maximizing his utility, because the Euler equation is still effective. This equation guarantees stability in this model and a smooth path to the new steady state. In other words, the stabilizing effect is played by the effort to equalize (discounted) marginal utilities over time, or alternatively to equalize the marginal rate of time preference with the ongoing real rate of interest.

[29] It should be stressed that even along the transition path, the natural rate of interest is in equilibrium. In the growth theory we have to distinguish between "non-steady state equilibria" and steady state equilibrium. From the static point of view, the first type of equilibrium is qualitatively the same as the second one. The key difference lies in the dynamic considerations.

probably too slow compared with the beliefs of the Austrian authors. Thus, a sudden increase in the level of technologies (e.g. a new invention) keeps the real natural rate of interest higher for a considerable period of time.

Furthermore, if the shock to the technological progress takes the form of an increase in the growth rate g, then the impact on the natural rate of interest is permanent. And this conclusion is at variance with Mises and Rothbard. The reason lies again in the fact that the Austrian authors neglected the influence of a varying income endowment on the rate of interest. In case of higher g, the income endowment should grow at a higher rate, thus the real rate of interest must be affected permanently.

An even more complicated behaviour of the interest rate is obtained if we allow for a stochastic element in the model. Consider an economy without permanent technological progress (i.e. g=0%) and with stationary population (n=0%). Suppose that the level of technologies follows a simple AR(1) or AR(2) process. The resulting behaviour not only of the natural rate of interest, but also of other most important variables is presented in Appendix 7, section H. However, the most interesting conclusion is that the natural interest is strongly affected by shocks to productivity.

In the previous sections, we stressed the fact that the analysis of interest must distinguish between the nominal approach and the real approach. The nominal approach is focused on the value difference between output and the expended inputs, whereas the centre of the real approach grounds in the exchange ratio between present goods and future goods. In the continuous infinite horizon model presented here, we have analysed the real natural rate of interest. However, the discussion of a simple model of the nominal interest rate from section ??? can be easily extended to this more complicated model.

The behaviour of the nominal rate of interest critically depends not only on the real interest rate but also on the evolution of prices. They are in turn affected by the development of output and the money supply. Here, we will assume again constant money supply and the velocity of circulation.[30] Thus, the behaviour of the price level will depend only on the growth rate of output. The exact size of the nominal interest rate can be found as the sum of the real interest rate and the inflation rate. Figures no. 21_A7 and 22_A7 in Appendix 7 demonstrate that when the economy is growing at a positive rate, the nominal rate of interest is lower than the real rate.

Furthermore, in the stochastic model, the nominal rate of interest seem to be much more volatile compared with the real interest rate, as it depends not only on that variable, but also on the growth rate of the economy (Figure no. 31_A7).

Figures no. 8_A7, 9_A7 (and 21_A7, 22_A7) and Figures no. 16_A7 and 17_A7 in Appendix 7 show the evolution of the nominal interest and real interest rate after the shock to the level (and the growth rate) of technology and the subjective discount rate. We have already seen that the steady state value of the real rate of interest might be zero (or negative) even in the RCK model. However, can this conclusion be applied also to the nominal rate of interest? In other words, can the value of output be permanently lower than the value of the expended inputs? A state that is absolutely unthinkable not only for the PTPT authors, but also (to a lower extent) for modern mainstream economics due to the belief in zero bound on nominal interest.

We will see that for constant money, the answer is definitely negative. In Appendix 7, we demonstrate that the Ramsey economy cannot be dynamically inefficient. At the steady state,

[30] This assumption obviously means that the (real/nominal) demand for money is homogenous of degree one in (real/nominal) income and it is independent of the (nominal) interest rate.

the growth rate of output can never exceed the real rate of interest. The technical reason lies in the fact that the lifetime utility must not diverge, i.e. ρ-n–(1-θ)g>0 by assumption. The steady state level of the real interest rate is $r^*=\rho + \theta g$, which is definitely higher than the BGP growth rate of output n+g. The proof is very simple: If ρ-n–(1-θ)g>0 then $\rho + \theta g > n+g$. Thus, $r^* > n + g$.

As a result, in the RCK model with constant money supply the nominal rate of interest is always positive, because the real rate of interest is higher than the rate of price deflation (n+g).[31] There is always a positive difference between the value of output and the value of expended inputs. The zero bound can never be hit. This holds not only at the BGP, but also at the entire saddle path. There might be an exception for the moment of a shock that suddenly and unexpectedly increases the growth rate of the economy. However, such a shock could not have been taken into account beforehand owing to its unpredictable nature, so the nominal rate of interest is most probably unaffected at that particular moment.

If we assume constant money and velocity, the nominal interest rate at the steady state in the RCK model can be precisely determined:

$$i^* = r^* + \pi \quad (44)$$

Since $r^* = \rho + \theta g$ and $\pi = - [dY(t)/dt]/Y(t) = - (n+g)$, (44) gives us:

$$i^* = \rho + \theta .\mathrm{g} - \mathrm{n} - \mathrm{g} \quad (45)$$

(45) might be written as:

$$i^* = \rho - n - (1-\theta)g > 0 \quad (46)$$

Thus, the nominal interest rate at steady state of the Ramsey model is positive even for constant money supply as long as the condition for the convergence of life-time utility holds. Surprisingly, it is even numerically equal to the specific combination of parameters required.

Furthermore, constant money, dynamically efficient economy and positive nominal interest rate are closely interconnected also in the RCK model. It is quite interesting that such a result is obtained again. Dynamic efficiency is in the first place a state that guarantees that consumption cannot be extended permanently without a sacrifice of consumption of some generation. However, it also leads to a positive nominal interest and a positive difference between the value of output and the value of the expended inputs. This property is quite unexpected and novel. It has been never mentioned by the PTPT authors or in the Austrian literature in general.

Moreover, equation (46) allows us to discuss whether changes in the growth rate of technologies will permanently affect the nominal interest rate, i.e. the value difference between output and the expended inputs. As can be seen, the answer critically depends on the value of the elasticity of substitution (1/θ). For constant money and velocity, increase in g may raise the nominal interest rate if this elasticity is rather low (high θ). On the other hand, relatively low preference for consumption smoothing results in a decrease in the steady state nominal interest rate after the rise in g (see Figure no. 22_A7 in Appendix 7). And finally, for

[31] Recall that for constant money (and velocity) the inflation rate π should be equal to the negative of the growth rate of output (which is n+g on the BGP).

logarithmic utility function, steady state nominal interest rate is not affected by the change in the rate of technological progress. Even though the transition period of its lower level seems to be significant, the Austrian pure time preference theory could be valid for this specific case, if it is defined as a theory of the value difference between output and expended inputs (i.e. in terms of the nominal interest rate) and not as a theory of the intertemporal exchange ratio among goods (i.e. in terms of the real interest rate). As has been seen in previous sections, only logarithmic utility function seems to be favourable to the Austrian PTPT. Our dynamic general equilibrium model just confirms this conclusion.[32]

Nevertheless, all these properties critically depend on the assumption of the convergence of the lifetime utility: ρ-n–(1-θ)g>0. At first glance, such a condition seems to be rather technical without any economic background. However, let us now present the economic foundation of this assumption. With zero population growth and stationary technology (or with changing technology and unitary elasticity of substitution, i.e. 1/θ =1) RCK model requires a positive subjective discount rate. This means that people must discount future utilities. Such a requirement is perfectly consistent with the Misesian maxim that people prefer the given satisfaction to be delivered as soon as possible.

If the size of a dynasty grows over time (n>0), the discount of future utilities must exceed the growth rate of the expansion of a family. However, even if the subjective discount rate is rather low (even zero), the important properties mentioned above might be obtained, if the technological progress is fast enough and people have "drastically diminishing marginal utility", i.e. θ>1. In other words, the non-divergence of the utility function, the dynamically efficient character of the model and the resulting positivity of the nominal interest are guaranteed, if the elasticity of substitution between present and future consumption is not very high. The economic explanation of that fact is as follows: Positive technological progress leads to an increasing time shape of the income stream. For a relatively high θ, people prefer a smoothed consumption path. Thus, higher income in the future compared with present leads to reduction in the saving rate (see the equation of the saving rate in Appendix 7, equation A7_79D).[33] As a result, the real rate of interest must grow. It will grow high enough to guarantee dynamic efficiency in the first place and a positive nominal rate of interest in the second place.

On the other hand, very high elasticity of substitution (i.e. low θ) is not consistent with this model, unless the subjective discount rate is high enough (ρ>>0) or the technological decay is fast enough (g<<0).[34] Thus, we can see that the analysis of the RCK model closely resembles the discussion of the discrete-time model with constant MPK in section ???. Under normal conditions with positive technological progress, a positive nominal rate of interest requires either a sufficient discount of future utilities or low elasticity of substitution.

Furthermore, in the text above we have shown that the real rate of interest in the RCK model might be zero on the balanced growth path even for a positive time preference (in sense two),

[32] Note that we assumed ZPG, i.e. n = 0 %, in this discussion.

[33] It can be perfectly seen, that the steady state optimum saving rate is negatively related to parameter θ provided that the technological progress is positive.

[34] Condition A7_14, ρ-n–(1-θ)g>0, requires θ < 1, if ρ=n=0 and g < 0. This means that if there is an economic decay, elasticity of substitution must be high enough. The economic reason for such a conclusion is as follows. Technological decline leads to a lower interest rate. For low enough θ, the saving curve is upward sloping, hence people do not save much for low interest rate. As a result, the economy will not over-accumulate capital, it is dynamically efficient and the nominal interest rate is positive (see section G in Appendix 7). On the other hand, if the preference for consumption smoothing was relatively high (θ >1), technological decline and the resulting expected fall in the income endowment would lead to excessive saving in the present. Thus, over-saving and dynamic inefficient character of the economy might emerge along with negative nominal interest rate. Hence, condition A7_14 prevents this possibility.

provided that $\rho=-\theta g$. Thus, to achieve zero real interest the technological progress must be negative at the rate of $g=-\rho/\theta$. Nevertheless, even for constant money the nominal rate of interest will be still positive, because the gradual fall in aggregate output will result in positive inflation rate driving up the nominal rate of interest above zero. A graphical representation of this process is given in Appendix 7, Section G, Figure no. 28_A7.

At the end of this paper, we will present characteristics that are most probably usual in normal times in the economy. First, the Misesian a-priori statement that people prefer the given satisfaction to be achieved as soon as possible was translated in a positive subjective discount rate ρ in our models. Its numerical value reflects the intensity of this preference. Second, the marginal productivity of capital must be decreasing at least from some point on the production function. And all profit maximizing (non-monopolistic) firms must operate beyond this point (Strigl ???).

If there was no technological progress, the economy should stabilize in such a state, in which the real rate of interest is positive and equal to ρ. If money was constant, then prices would be stable and the nominal rate of interest would coincide with the real rate of interest. Thus, the nominal interest would be definitely positive as well as the value difference between output and the expended inputs. In such a state, the Austrian analysis would be almost indistinguishable from the usual neoclassical theory, since the Misesian idea of ERE closely resembles that of stationary equilibrium. (Figures ?).

However, the picture that is closest to real world is more consistent with positive technological progress. In such a case, the economy should sooner or later find its balanced growth path. At this state, the real rate of interest depends not only on the subjective discount rate, but also on the growth rate of income and the intertemporal elasticity of substitution of consumption. However, it is definitely positive. So “present goods are valued more than future goods of the same kind and number.” If money is constant, prices must naturally fall. However, the nominal rate of interest along with the value difference between output and inputs are positive, because the economy is certainly dynamically efficient (real interest rate is higher than the growth rate in GDP). Yet, its numerical value should be lower than that of the real rate due to the “secular deflation of the price level”. Even though it might be unthinkable for the majority of economists that prices should fall, the author of this paper strongly believes that this should be a normal state of a prosperous economy in a dynamic general equilibrium.

Figures

36, 37 Missing

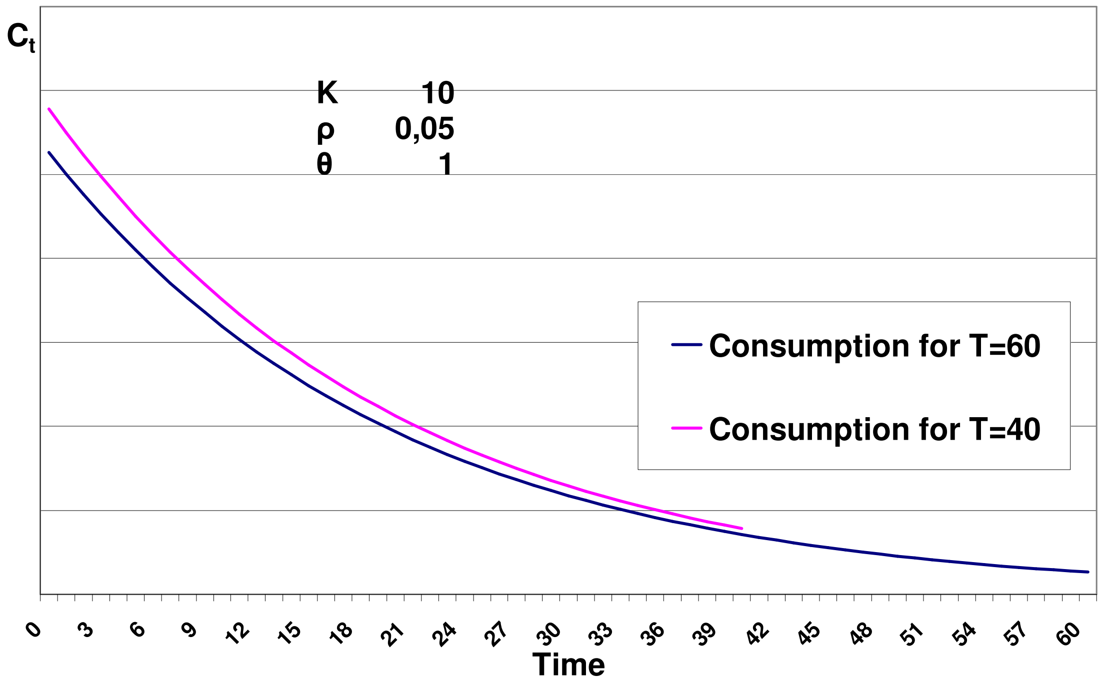


Figure no. 38 Optimum path of consumption in a hard-tack economy for varying time horizon. θ=1, ρ=0.05

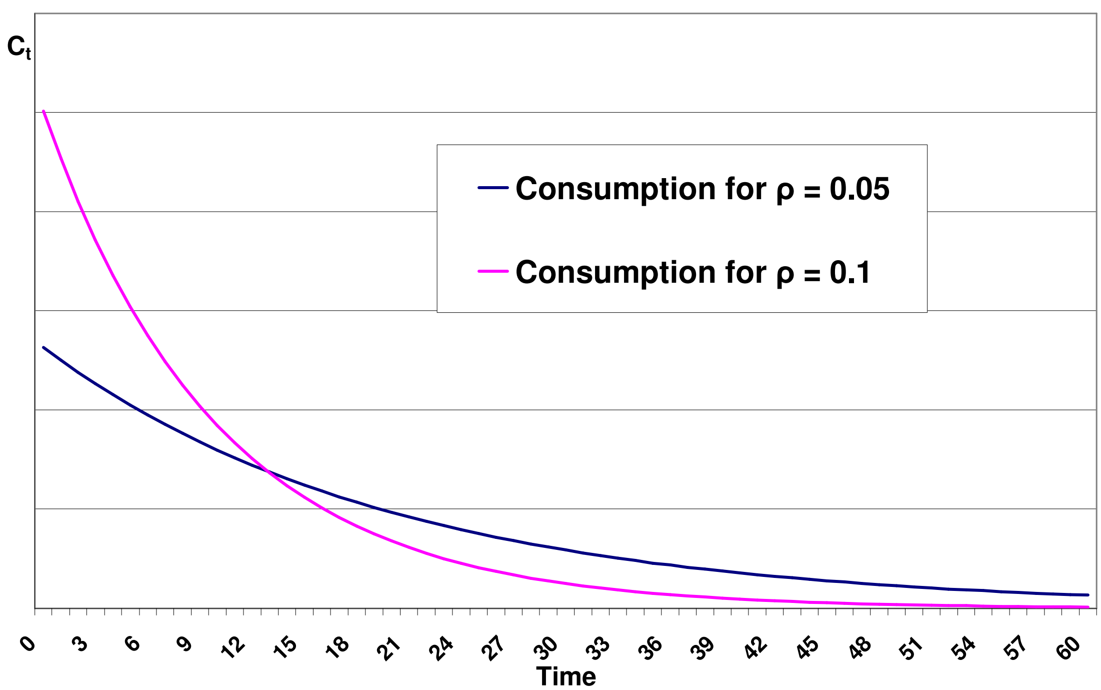


Figure no. 39 Optimum path of consumption in a hard-tack economy for varying subjective discount rate; θ=1.

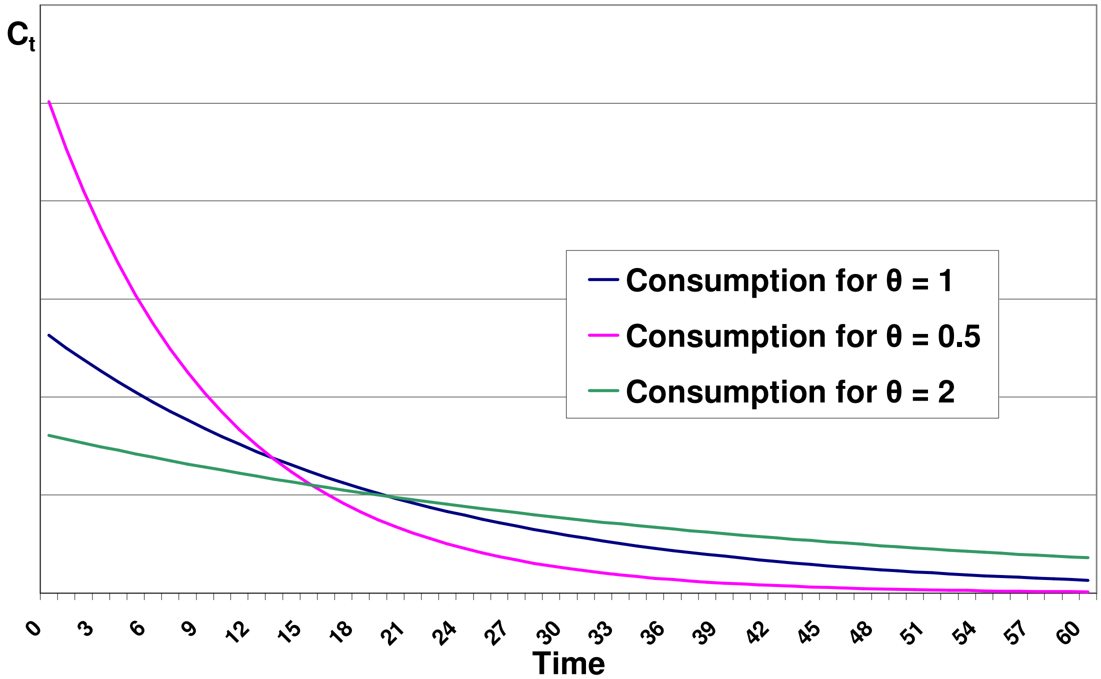


Figure no. 40 Optimum path of consumption in a hard-tack economy for varying elasticity of substitution (1/θ); ρ = 0.05.

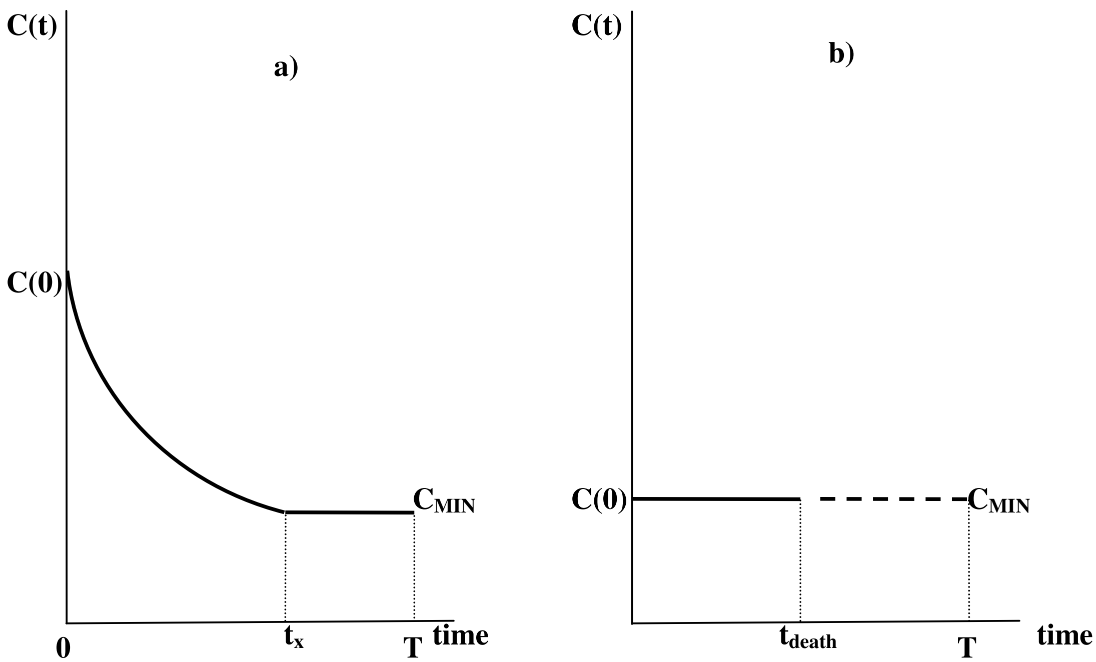


Figure no. 41 Optimum consumption path if the subsistence level is achieved within the planning horizon.

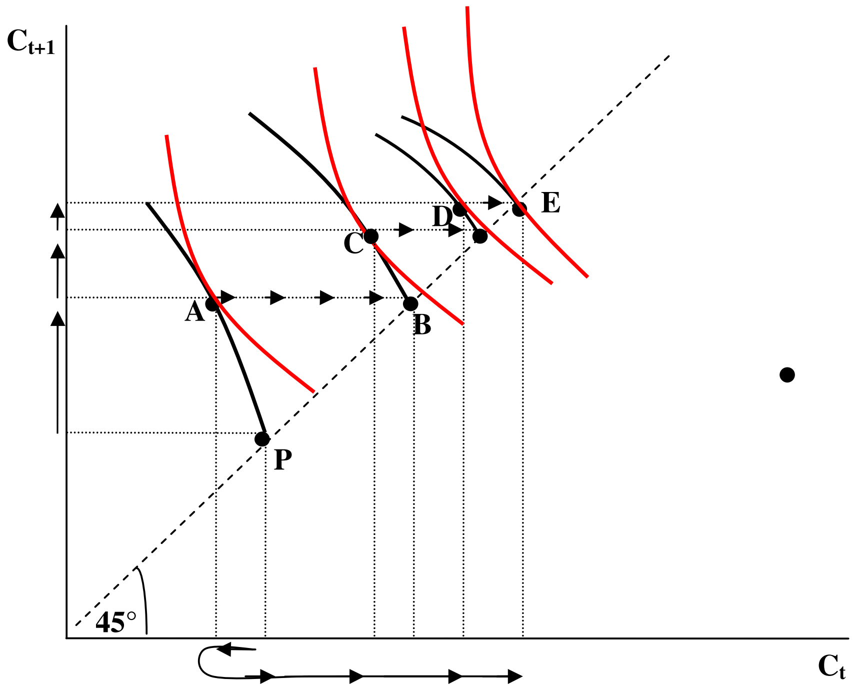


Figure no. 42 Hayek´s representation of the process of the accumulation of capital (1941:???).

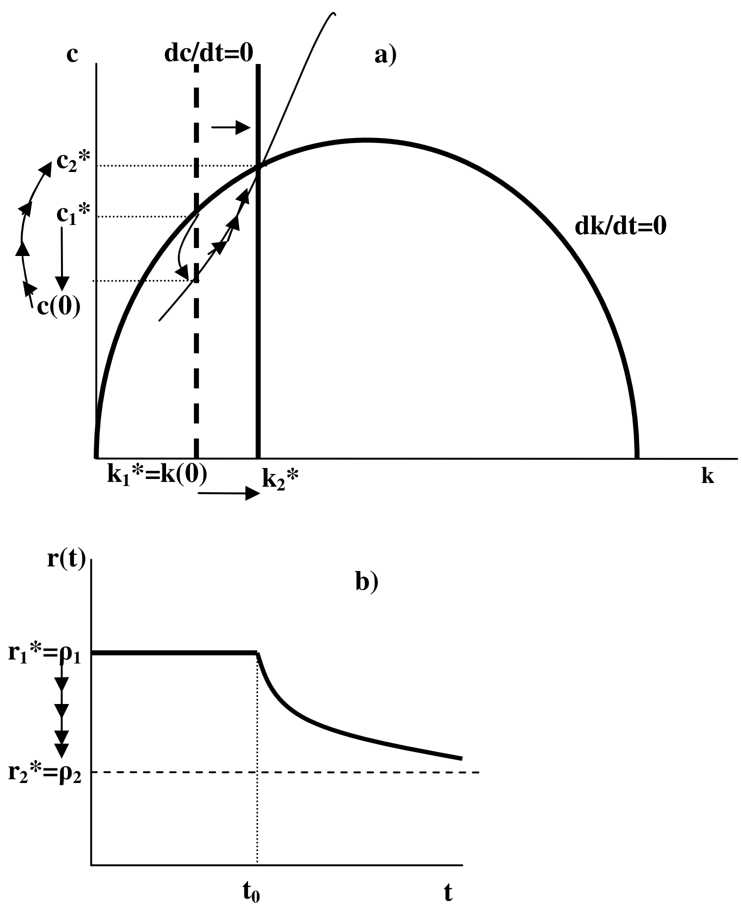


Figure no. 43 Convergence of the natural rate of interest in the RCK model after a decrease in ρ.

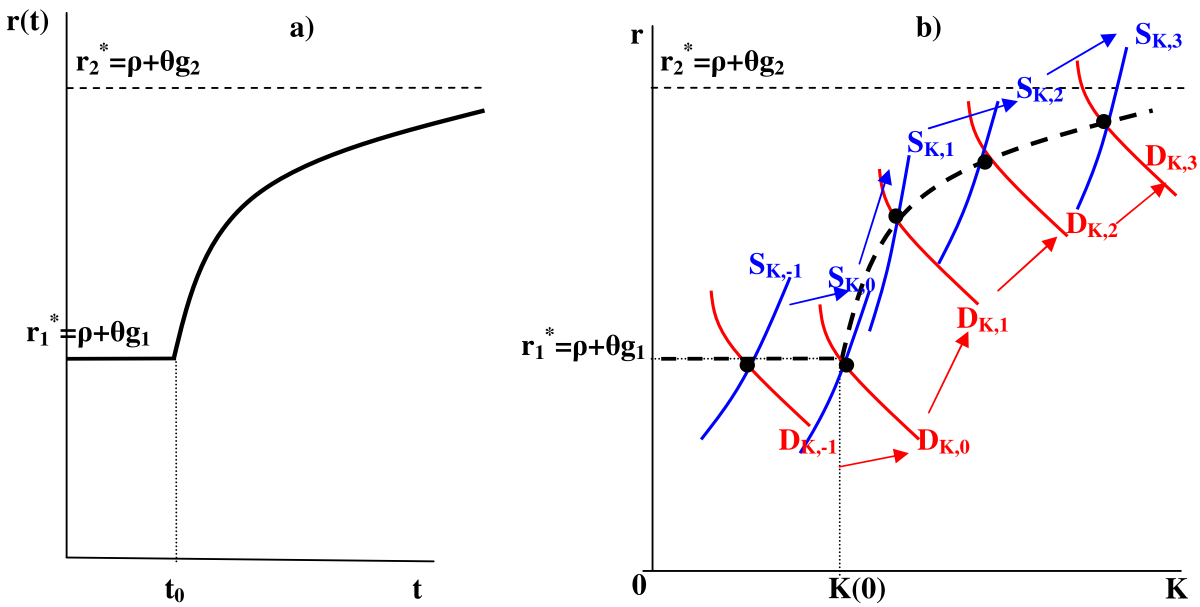


Figure no. 44 Convergence of the natural rate of interest in the RCK model after a sudden increase in the growth rate of technological progress.

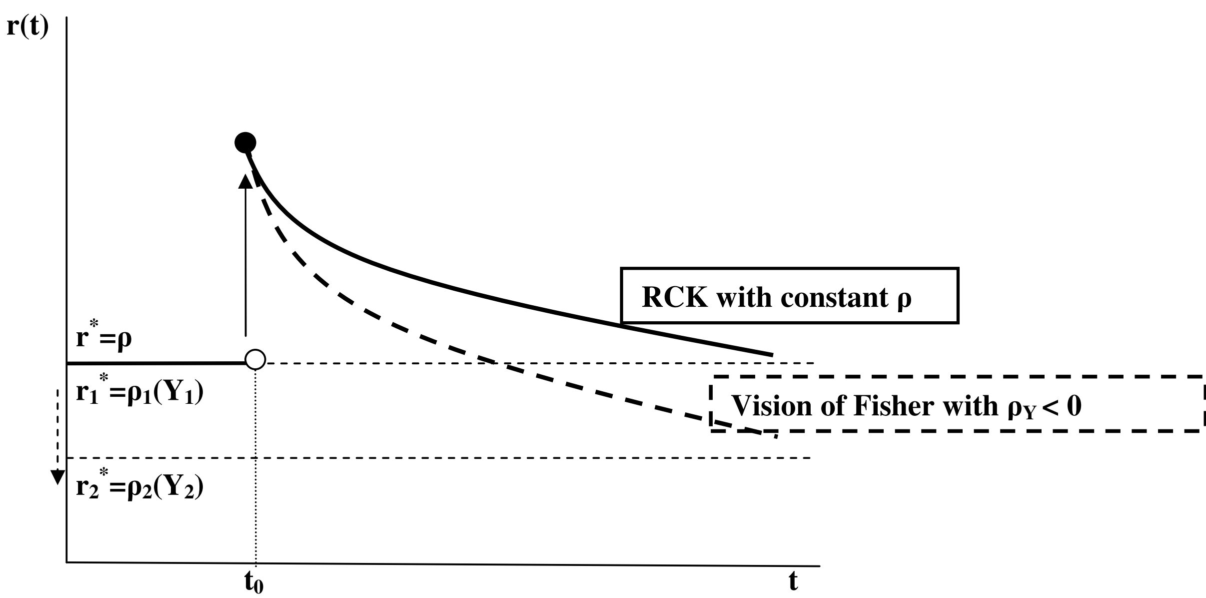


Figure no. 45 Evolution of the natural rate of interest in the RCK model after a sudden increase in the level of technologies A. The dashed line represents prediction of I. Fisher.

Appendix 4

In this Appendix, we will solve the optimization problem of a representative consumer from section 5. As usual, his objective is to find the optimum path of consumption so as to maximize his lifetime utility (A4_1), subject to his lifetime intertemporal budget constraint (A4_2).[35]

$$U = \sum_{t=0}^{T} \frac{u(C_t)}{(1+\rho)^t} = u(C_0) + \frac{u(C_1)}{1+\rho} + \frac{u(C_2)}{(1+\rho)^2} + \ldots + \frac{u(C_T)}{(1+\rho)^T} \qquad \text{(A4\_1)}$$

$$\mathrm{C}_0 + \frac{\mathrm{C}_1}{(1+\mathrm{r})} + \frac{\mathrm{C}_2}{(1+\mathrm{r})^2} + \ldots + \frac{\mathrm{C}_T}{(1+\mathrm{r})^{\mathrm{T}}} = \mathrm{Y}_0 + \frac{\mathrm{Y}_1}{(1+\mathrm{r})} + \frac{\mathrm{Y}_2}{(1+\mathrm{r})^2} + \ldots + \frac{\mathrm{Y}_T}{(1+\mathrm{r})^{\mathrm{T}}} \qquad \text{(A4\_2)}$$

Set up a simple Lagrangian function for this problem:

$$L = u(C_0) + \frac{u(C_1)}{1+\rho} + \frac{u(C_2)}{(1+\rho)^2} + \ldots + \frac{u(C_T)}{(1+\rho)^T} + \lambda(Y_0 + \frac{Y_1}{1+r} + \frac{Y_2}{(1+r)^2} + \ldots + \frac{1}{(1+r)^T} Y_T -$$

$$C_0 - \frac{1}{1+r} C_1 - \frac{1}{(1+r)^2} C_2 - \ldots - \frac{1}{(1+r)^T} C_T) \qquad \text{(A4\_3)}$$

Let us find the first order conditions for optimum consumption at any time t and t+1:

FOC:

$$\frac{\partial L}{\partial C_t} = \frac{u'(C_t)}{(1+\rho)^t} - \lambda \frac{1}{(1+r)^t} = 0 \qquad \text{(A4\_4)}$$

$$\frac{\partial L}{\partial C_{t+1}} = \frac{u'(C_{t+1})}{(1+\rho)^{t+1}} - \lambda \frac{1}{(1+r)^{t+1}} = 0 \qquad \text{(A4\_5)}$$

Expressing λ in both periods and dividing (A4_5) by (A4_4) we get:

$$\frac{u'(C_{t+1})}{u'(C_t)} = \frac{1+\rho}{1+r} \qquad \text{(A4\_6)}$$

(A4_6) is the Euler equation for this problem. Alternatively, FOC for C_0 is just $\lambda = u'(C_0)$. Thus, using (A4_4) the Euler equation might be written as:

$$\frac{u'(C_t)}{u'(C_0)} = \left(\frac{1+\rho}{1+r}\right)^t \qquad \text{(A4\_7)}$$

[35] This IBC for constant income is assumed in Olson, Bailey (1981:9). Its derivation is presented in Appendix 5. For a better understanding, the reader is advised to read Appendix 5 first.

Applying a specific CRRA utility function, for which the marginal utility of consumption is $C^{-\theta}$, (A4_6) yields (see equation 19 in the main text):

$$\frac{C_t}{C_{t+1}} = \left(\frac{1+\rho}{1+r}\right)^{1/\theta} \qquad \text{(A4\_8)}$$

Similarly, (A4_7) is modified to:

$$\frac{C_0}{C_t} = \left(\frac{1+\rho}{1+r}\right)^{t/\theta} \qquad \text{(A4\_9)}$$

(A4_9) might be used to solve the initial optimal level of consumption and then consumption at any time. Assuming θ=1 (i.e. logarithmic utility function), (A4_9) for time 0 and 1 is:

$$C_1 = \frac{1+r}{1+\rho} C_0 \qquad \text{(A4\_10)}$$

(A4_8) and (A4_9) for time 1 and 2 are:

$$C_2 = \frac{1+r}{1+\rho} C_1 = \left(\frac{1+r}{1+\rho}\right)^2 C_0 \qquad \text{(A4\_11)}$$

Substituting C_1, C_2 etc. into the intertemporal budget constraint (A4_2) and for infinite T, we may write:

$$\mathrm{C}_0 + \frac{\frac{1+r}{1+\rho} C_0}{(1+\mathrm{r})} + \frac{\left(\frac{1+r}{1+\rho}\right)^2 C_0}{(1+\mathrm{r})^2} + \ldots = \mathrm{Y}_0 + \frac{\mathrm{Y}_1}{(1+\mathrm{r})} + \frac{\mathrm{Y}_2}{(1+\mathrm{r})^2} + \ldots \qquad \text{(A4\_12)}$$

$$\mathrm{C}_0 + \frac{\mathrm{C}_0}{1+\rho} + \frac{\mathrm{C}_0}{(1+\rho)^2} + \ldots = \mathrm{Y}_0 + \frac{\mathrm{Y}_1}{(1+\mathrm{r})} + \frac{\mathrm{Y}_2}{(1+\mathrm{r})^2} + \ldots \qquad \text{(A4\_13)}$$

The right-hand side of (A4_13) represents the present value of the flow of income (PV_{income}). Using the formula for the sum of the infinite geometric series, (A4_13) becomes:

$$\frac{\mathrm{C}_0}{1-\frac{1}{1+\rho}} = PV_{income} \qquad \text{(A4\_14)}$$

$$\mathrm{C}_0 = \frac{\rho}{1+\rho} PV_{income} \qquad \text{(A4\_15)}$$

(A4_15) and (A4_9) might be used to derive consumption in any period. For time 1, 2 and t (see A4_10 and A4_11), we get:

$$C_1 = \frac{\rho(1+r)}{(1+\rho)^2} PV_{income} \quad \text{(A4\_16)}$$

$$C_2 = \frac{\rho(1+r)^2}{(1+\rho)^3} PV_{income} \quad \text{(A4\_17)}$$

$$C_t = \frac{\rho(1+r)^t}{(1+\rho)^{t+1}} PV_{income} \quad \text{(A4\_18)}$$

For a zero time preference in sense two (ρ=0), i.e. if people do not prefer the given want to be gratified as soon as possible, a man will not consume in the present. As can be seen in (A4_15), C_0=0. But he will not consume in the next period either. Moreover, he will not consume in any future period, expect for infinity. Consumption will be postponed forever. Thus, it seems that Mises was right that with zero time preference and positive interest rate, the act of consumption will never occur.

Yet, if we look at (A4_15) again, C_0 might be non-zero (or more precisely it may take on any value), if PV of income is infinite. This might occur either if the real interest rate declines to zero and the flow of income is sufficiently non-decreasing over time, or if the growth rate in income exceeds (or is equal to) the real interest rate. However, the later statement would indicate a dynamic inefficiency. Thus, we will exclude the possibility of an infinite present value of the income stream.

Olson and Bailey (???:???) suggested that zero time preference is consistent with positive real interest rate and positive present consumption, if the marginal utility of consumption dramatically decreases with higher consumption and the labour income endowment grows over time. (A4_15) and (A4_18) are derived for θ=1, i.e. for a logarithmic utility function. As we will see below, θ=1 might be too low to generate such a property.

As a result, let us try to recalculate the key equations for any value of θ. However, it might be more instructive to start with a finite horizon to grasp the key idea. Thus, consider the budget constraint (A4_2). If the income process obeys the following process $Y_t=(1+g)Y_{t-1}$, the present value of the income stream is given by (see A5_24 in Appendix 5):

$$PV_{income} = \frac{(1+r)^{T+1} - (1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_0 \quad \text{(A4\_19)}$$

Using the equation for the optimal flow of consumption (A4_9), the left hand side of the intertemporal budget constraint (A4_2) might be written as:

$$PV_{consumption} = C_0 + \frac{C_0\left(\frac{1+r}{1+\rho}\right)^{1/\theta}}{1+r} + \frac{C_0\left(\frac{1+r}{1+\rho}\right)^{2/\theta}}{(1+r)^2} + ... + \frac{C_0\left(\frac{1+r}{1+\rho}\right)^{T/\theta}}{(1+r)^T} \quad \text{(A4\_20)}$$

$$PV_{consumption} = C_0 + C_0 \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} + C_0\left[\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]^2 + ... + C_0\left[\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]^T \quad \text{(A4\_21)}$$

According to the formula of the sum of the finite geometric sequence, we may write:

$$PV_{consumption} = C_0 \frac{1 - \left[\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]^{T+1}}{1 - \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}} \qquad \text{(A4\_22)}$$

$$PV_{consumption} = \frac{(1+\rho)^{(T+1)/\theta} - (1+r)^{(T+1)(1-\theta)/\theta}}{(1+\rho)^{(T+1)/\theta}} \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}} C_0 \qquad \text{(A4\_23)}$$

From (A4_19), (A4_23) and (A4_2) we can easily determine the optimal level of present consumption C_0:

$$PV_{consumption} = PV_{income} \quad \text{(A4\_24)}$$

$$\frac{(1+\rho)^{(T+1)/\theta} - (1+r)^{(T+1)(1-\theta)/\theta}}{(1+\rho)^{(T+1)/\theta}} \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}} C_0 = \frac{(1+r)^{T+1} - (1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_0$$

(A4_25)

$$C_0 = \frac{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} \frac{(1+\rho)^{(T+1)/\theta}}{(1+\rho)^{(T+1)/\theta} - (1+r)^{(T+1)(1-\theta)/\theta}} \frac{(1+r)^{T+1} - (1+g)^{T+1}}{(1+r)^{T+1}} \frac{1+r}{1-g} Y_0$$

(A4_26)

$$C_0 = PV_{income} \frac{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} \frac{(1+\rho)^{(T+1)/\theta}}{(1+\rho)^{(T+1)/\theta} - (1+r)^{(T+1)(1-\theta)/\theta}} \qquad \text{(A4\_27)}$$

Using (A4_9), the value of consumption in any period might be then determined. As can be seen, the present consumption depends on the interest rate, time preference in sense two (i.e. the subjective discount rate), the elasticity of substitution 1/θ, the time horizon T and the present value of the income stream, which in turn depends on the initial income, interest rate, and the growth rate in income g.

According to Olson and Bailey, the present consumption in the infinite horizon model must be depressed to zero, if the time preference is zero, unless income endowment grows over time and people have a drastically diminishing marginal utility. Before analyzing an infinite horizon model, let us use our finite horizon model to explore their argument.

We introduce a set of various simulations. In the first section, we assume a positive growth in the income endowment at the rate of g = 1 %. This rate was assumed in Olson and Bailey (???:???).

App. 4 simulations

Appendix 4 Simulations

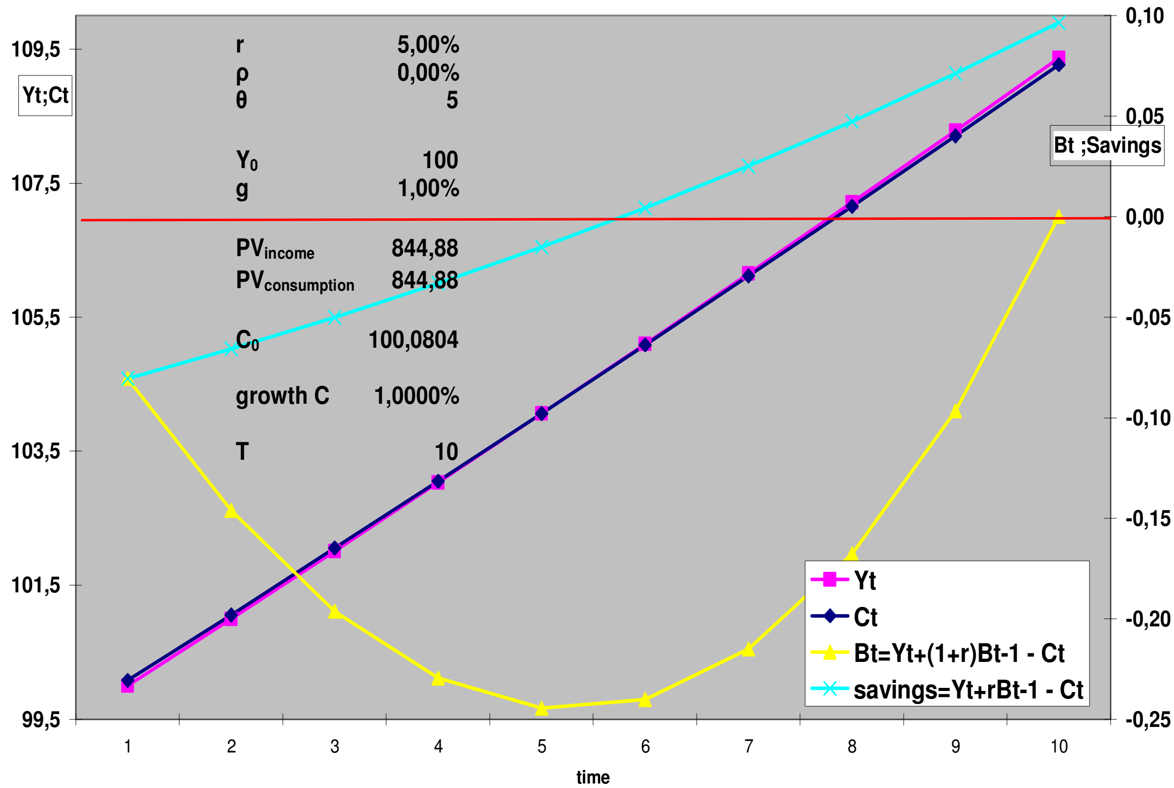


Figure no. 1_A4

Figure no. 1_A4 is designed for real interest rate r = 5 %, initial labour income Y_0=100, its growth rate of 1 %, zero time preference (ρ = 0) and relatively high θ=5, which was assumed by Olson,Bailey (???:???)Thus, for such a high θ, the marginal utility is surely drastically diminishing. We start with a relatively short time horizon, only 10 years.

Our model predicts that the optimal present consumption is slightly above the initial income level. By no means, it is depressed to a zero level. It grows at the optimal rate of 1 %. Because consumption exceeds income in the initial periods, saving is negative.[36] Hence, debt is issued ($B_0 < 0$) and it is gradually accumulated. However, due to the No-Ponzi-Game condition imposed on our model (see Appendix 5 for a thorough discussion), its terminal value must be zero, i.e. $B_{t=T=10}$=0 (see the final point of the yellow curve). As can be seen, in period 6 savings are positive and debt might be reduced, as it reaches its maximum in period 5.

Figure no. 2_A4 extends the time horizon to 50 years. Nevertheless, similar conclusions might be said here as in Figure no.1_A4. The same holds for time horizons 100 years and 500 years. In neither case, present consumption is depressed to zero, even though people do not prefer the given satisfaction to be delivered as soon as possible (i.e. they do not discount future utilities, ρ = 0 %). The reason for a non-zero present consumption might be an increasing profile of their income stream and a dramatically decreasing MU, so the analysis of Bailey and Olson seems to be accurate. Future is better provided for and people prefer relatively smoothed profile of consumption, which results in a positive premium on present goods, whose consumption thus cannot fall to zero. Future higher income is therefore moved closer to the present and saving is negative in the first part of the relevant time horizon.

[36] As will be seen below, this outcome is due to the approximation made by Olson and Bailey.

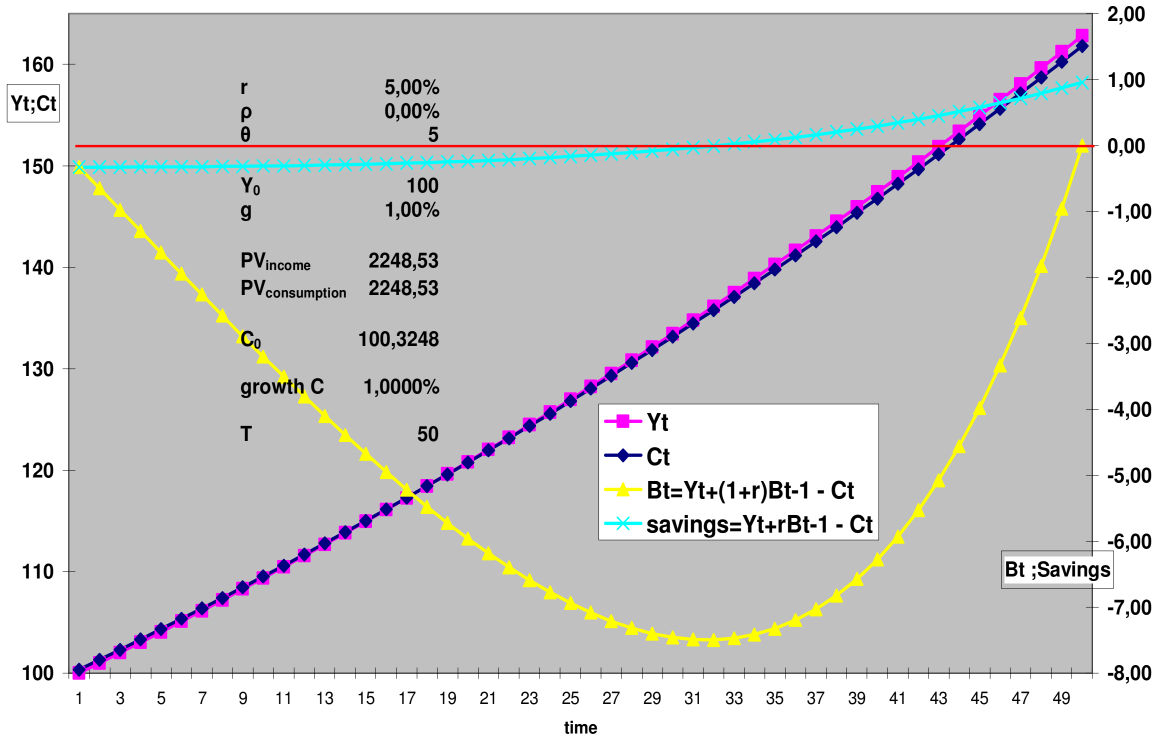


Figure no. 2_A4

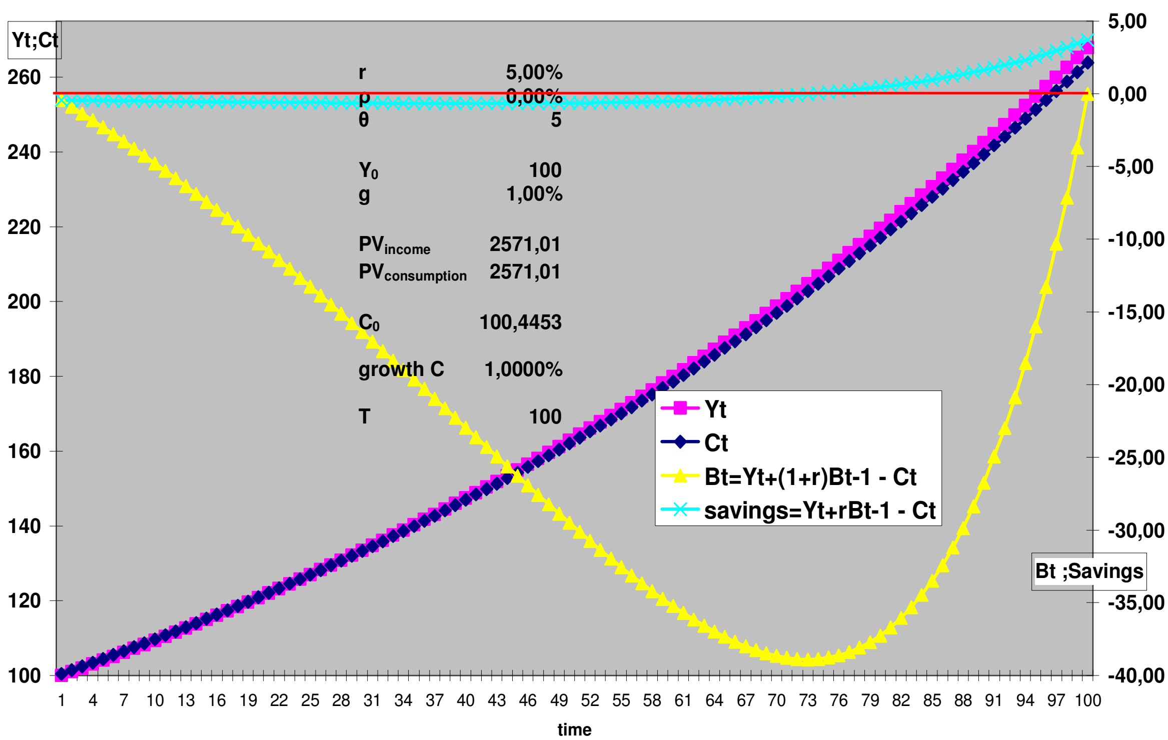


Figure no. 3_A4

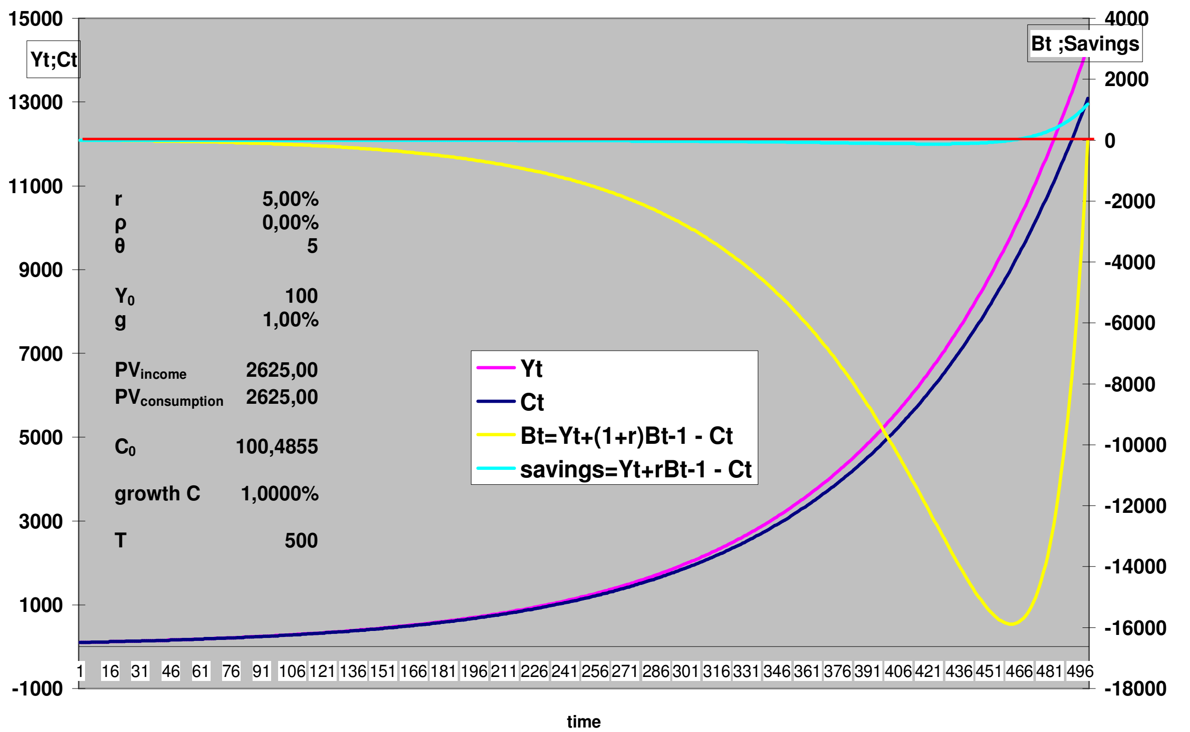


Figure no. 4_A4

In the next part, let us reduce parameter θ and thus increase the elasticity of substitution. Keeping all the other parameters at the same level, optimum consumption growth is then not in accordance with the exogenous income growth; it is higher. Figure no. 5_A4 represents an individual with θ=2 and 100-year planning horizon.

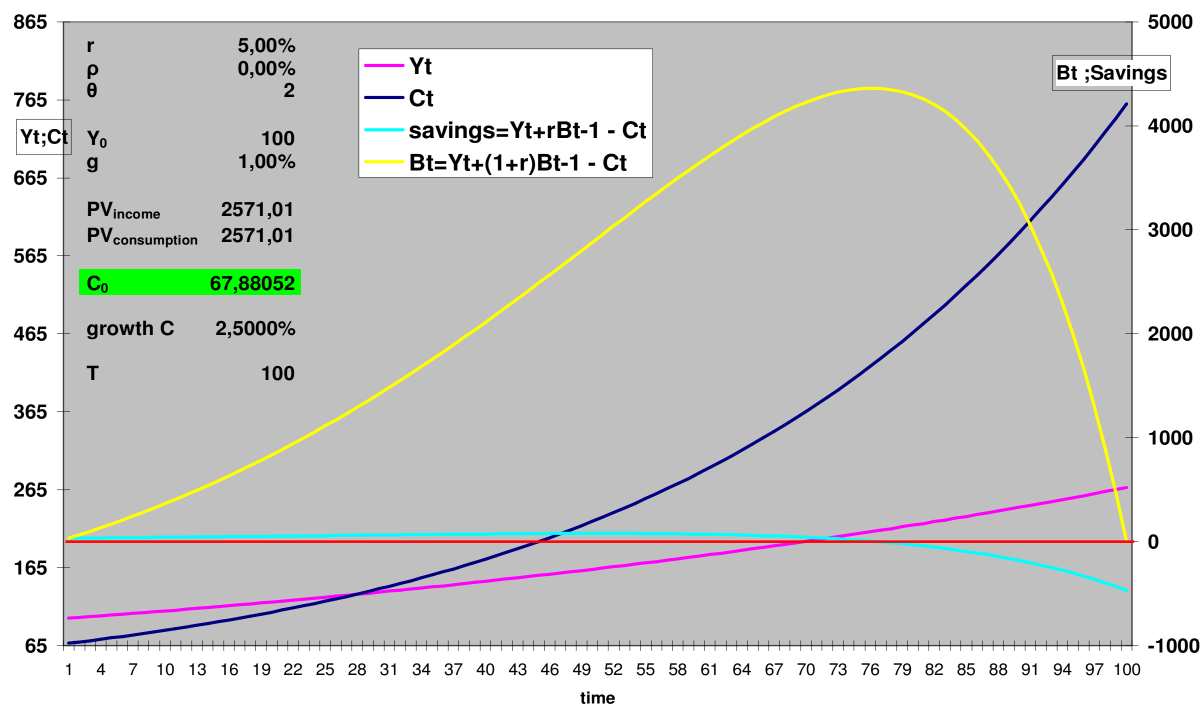


Figure no. 5_A4

As can be seen in Figure no. 5_A4, higher elasticity of substitution (i.e. less dramatically decreasing MU) reduces present consumption to 68 % of present income. As a result, saving is positive for a considerable part of the planning horizon and assets are accumulated to relatively high levels, which enables the future consumption to exceed future labour income by a large amount. This point is of crucial importance and it will be stressed in further sections again. Long planning horizons lead to the fact that compounded interest allows for very high future consumption levels quite independent of the future levels of the labour income. Thus, it seems that an increasing profile of the labour income stream might not be the crucial reason for non-zero present consumption, if, at the same time, future consumption is to reach very high levels.

Figure no. 6_A4 displays simulation of logarithmic utility function (i.e. $\theta \rightarrow 1$) and a planning horizon of 50 years. Longer horizons would give us an analogous picture, but of an inferior clarity. As can be seen, similar conclusions might be derived as in the previous case. Yet, present consumption is even more depressed to zero for the benefit of future consumption. Thus, we may conclude that the higher the elasticity of substitution (i.e. the lower the parameter θ), the lower the optimum present consumption, the higher the optimal growth rate in consumption and the higher the optimum future consumption, whose astronomical future values are quite independent of future-period labour income.

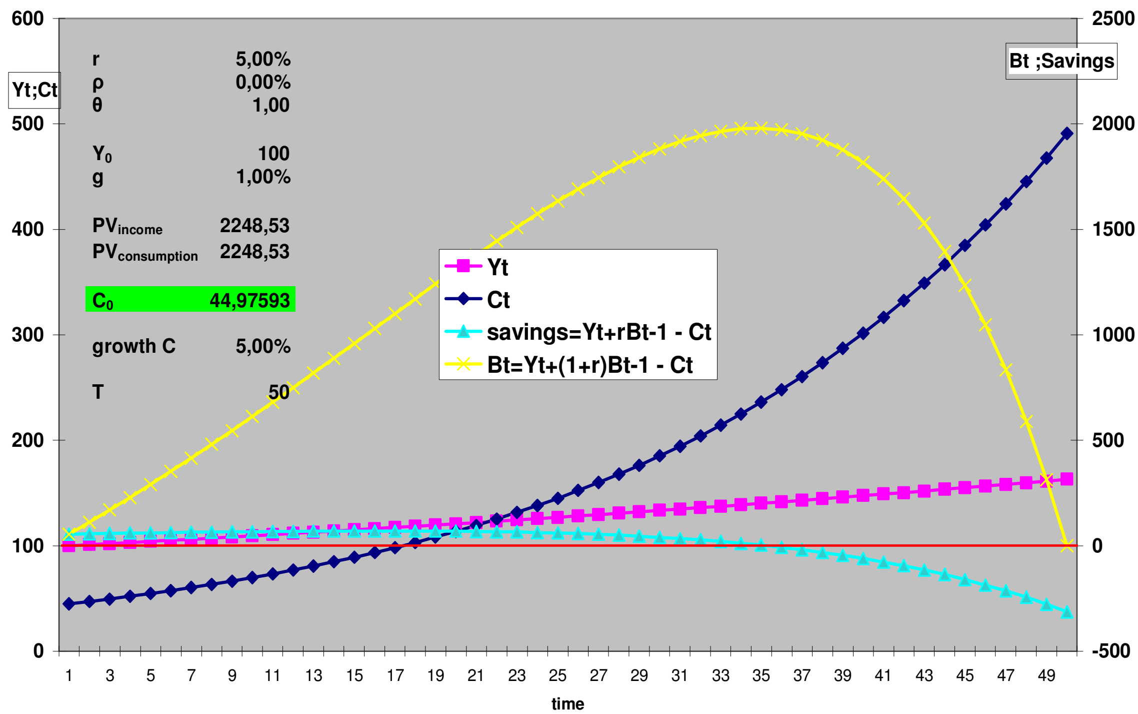


Figure no. 6_A4

Nonetheless, exactly zero present consumption and postponement of all consumption to the future, as was predicted by Mises for zero time preference, might be achieved only for the case of perfect substitutes (i.e. $\theta \rightarrow 0$). This can be seen in Figure no. 7_A4, which is designed for only a 20-year horizon. On the other hand, our analysis also suggests that longer time horizons might also considerably depress present consumption provided that θ is quite low but not necessarily zero.[37] Thus, an infinite horizon model might give the Misesian analysis some credit.

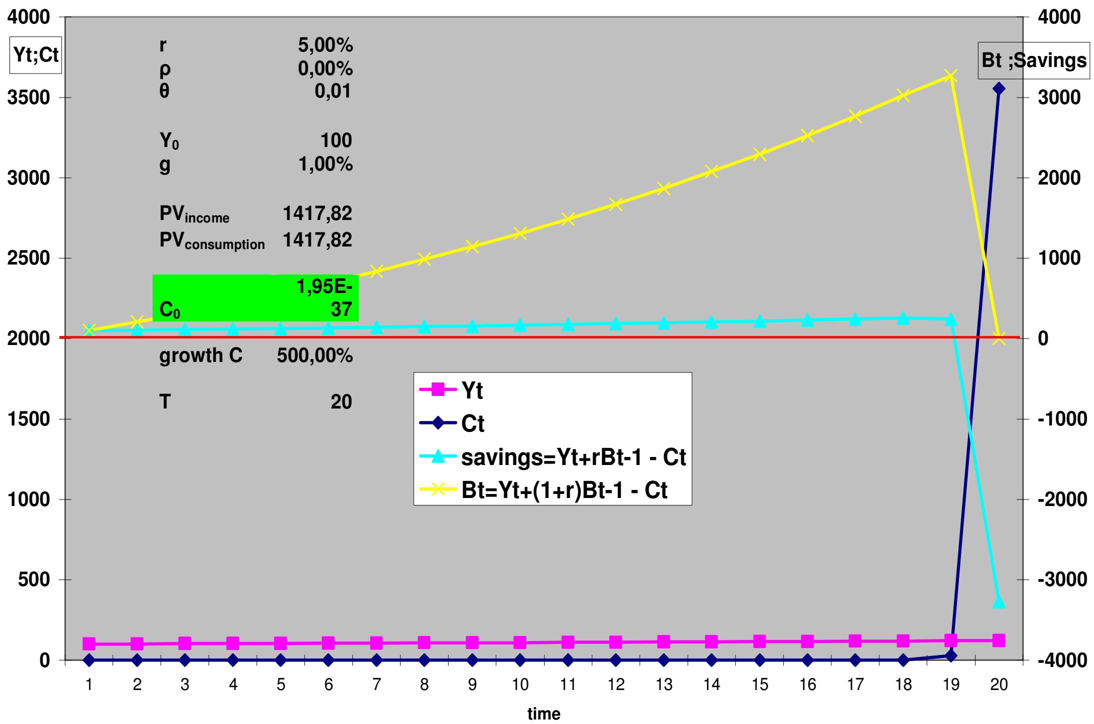


Figure no. 7_A4

In this section, we relax the assumption of an increasing income, thus we set g=0%. Olson and Bailey (???:???) suggested that if time preference is zero, constant income stream must lead to zero present consumption, if the planning horizon is infinite. Before we test their prediction, let us examine a finite horizon model for the same set of assumptions, i.e. zero income growth and zero time preference.

Figure no. 8_A4 is designed for real interest rate r = 5 %, constant labour income stream of Y_0=100, zero time preference (ρ=0%) and θ = 5, which was assumed by Olson,Bailey (???:???).We start again with a relatively short time horizon of 10 years. As can be seen in this figure, the optimum growth rate of consumption is 1 % as in Figure no. 1_A4, since it depends on the real interest rate, subjective discount rate and the elasticity of substitution, not on the specific shape of the income stream. However, the constancy of income requires that present consumption must be below the present labour income to generate sufficient growth in consumption, because the level of future consumption must exceed the level of future income, if the entire lifetime income is to be completely exhausted.

As a result, saving is positive in the first part of the planning horizon and assets are being accumulated. The accrued interest in a 10-year horizon is quite modest so the future consumption is quite close to future labour income. The stock of assets (i.e. B_t) reaches its maximum in period 5, as from period 6 saving is negative and assets gradually decline to a

[37] E.g. for θ=0.7 and T=50 years, optimum present consumption is about 25. For T=200 years it is depressed to 0.86.

zero level. As is stressed in Appendix 5, for a monotonically increasing utility function it cannot be optimal to hold any positive assets at the end of the planning horizon.

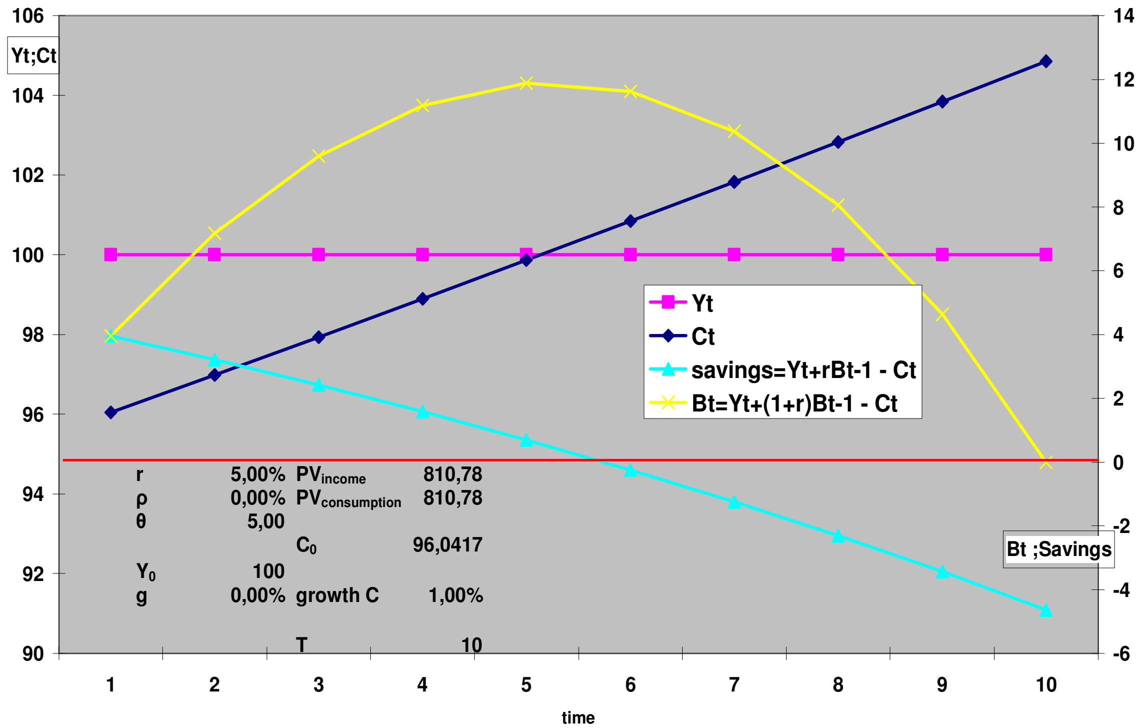


Figure no. 8_A4

A similar picture can be deduced from longer time horizons (see Figures no. 9_A4, 10_A4, 11_A4, 12_A4).[38] The only difference is a relatively larger reduction in the present consumption compensated by a higher level of future consumption. However, even a very long time horizon of 500 years will not depress the present consumption to zero, although more assets must be accumulated in the first part of the planning horizon. The reason is a relatively low elasticity of substitution that results in a very modest optimum growth rate in consumption that is below the real interest rate. The initial accumulation of assets then provides enough capital income to finance very high levels of future consumption that will considerably exceed future-period labour income. Thus, it seems that an increasing profile in labour income is not necessary as was predicted by Olson and Bailey (1981). A positive difference between the real interest rate and the growth rate in consumption and sufficient saving at the beginning of the planning horizon suffice to generate very high levels of future consumption even without the reduction of present consumption to zero.[39]

Another interesting observation, discussed in Appendix 5 in a more detail, is that saving might be positive, even if the given period consumption (C_t) is greater than the given period labour income (Y_t). The reason lies in the fact that the relevant income source for saving is the

[38] In the 500-year planning horizon, the development of crucial variables was separated in more graphs for greater clarity. As can be seen in Figure no. 11_A4, the eventual future consumption is more than 100 times higher than the labour income. Furthermore, the first 100 years of this planning horizon were also reported separately to track the initial accumulation of assets.
[39] It can be perfectly seen (especially in Figures 12,13 and 14) that relatively modest savings every period (a flow concept) might generate huge assets (stock concept) over a very long time period.

disposable income ($Y_t + r_1B_{t-1}$), i.e. the sum of labour income and capital income, not the simple labour income. Until the difference between the disposable income and consumption is positive, the stock of assets might be increasing.[40]

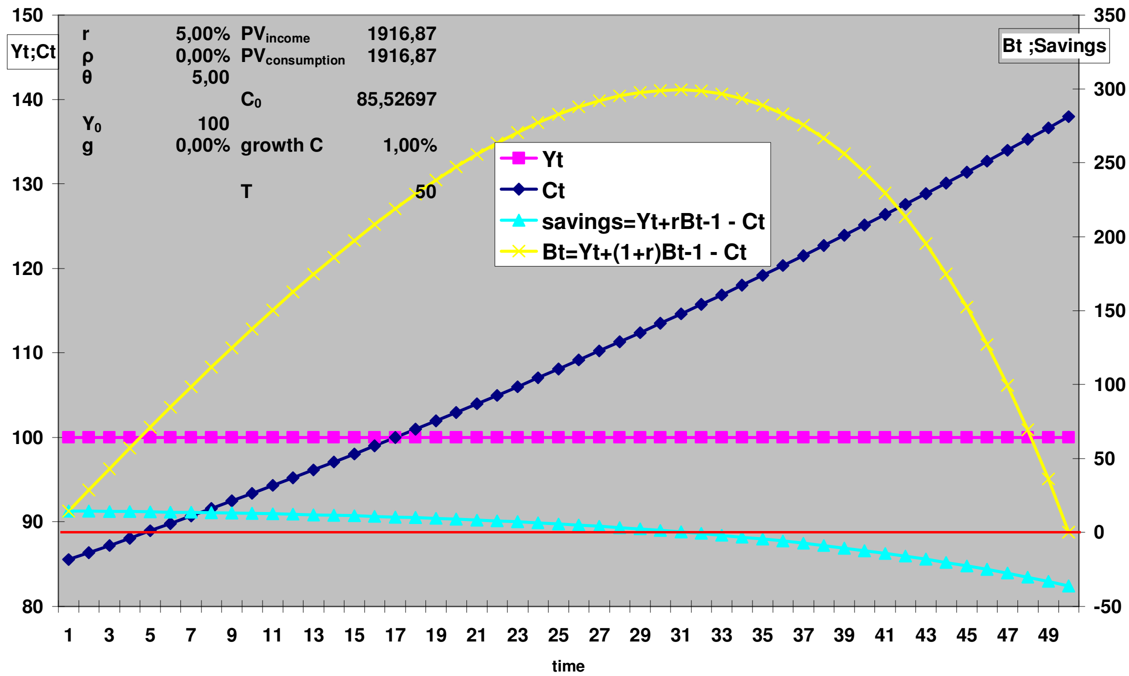
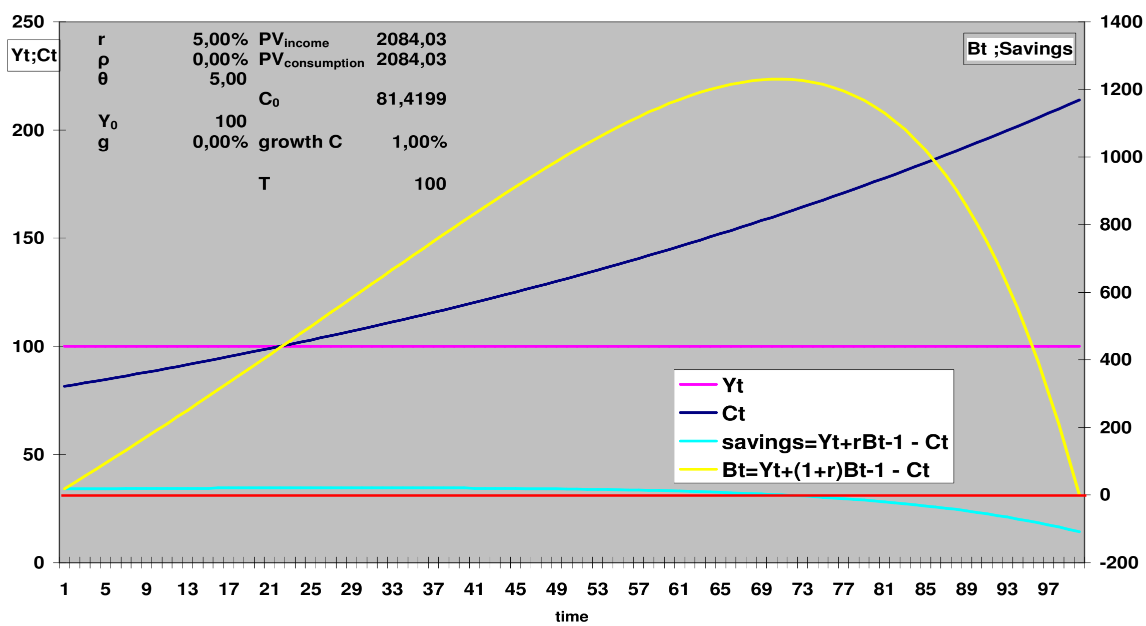


Figure no. 9_A4



[40] In the balance of payments jargon, the difference between Y and C might represent the trade balance, the difference between the disposable income and consumption then the entire current account reporting the sum of the trade balance and the income balance rB. Thus, the current account might be positive (and the financial account therefore negative indicating a positive net outflow of capital, i.e. positive net foreign investment), even if the trade balance is negative provided that the income balance reports a sufficient surplus.

Figure no. 10_A4

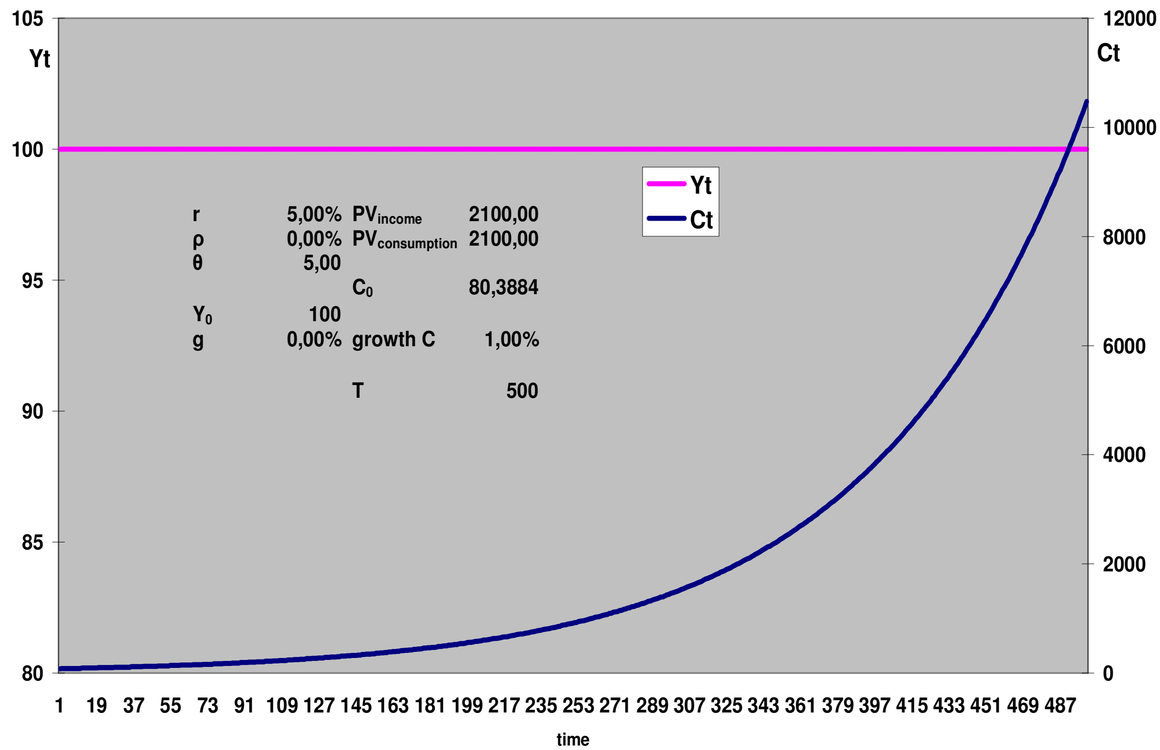


Figure no. 11_A4

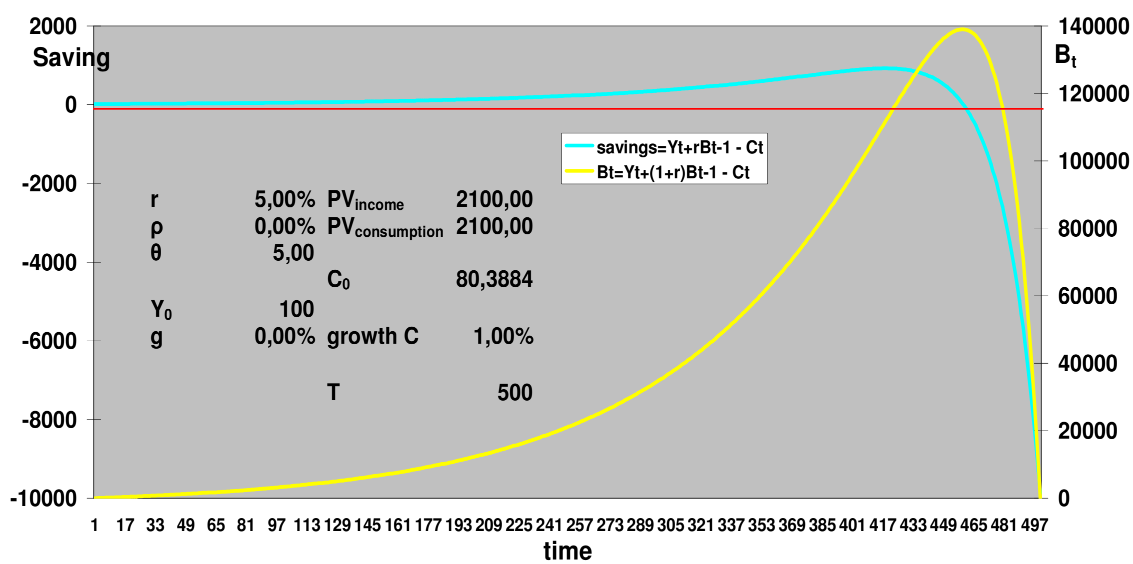


Figure no. 12_A4

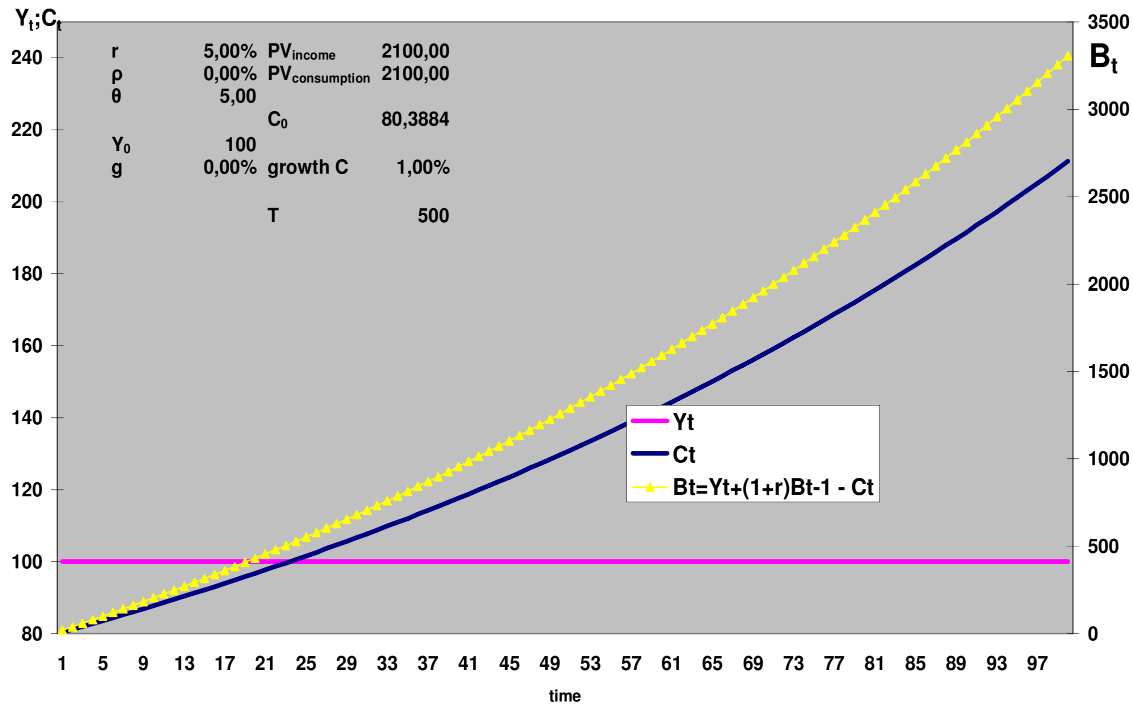


Figure no. 13_A4

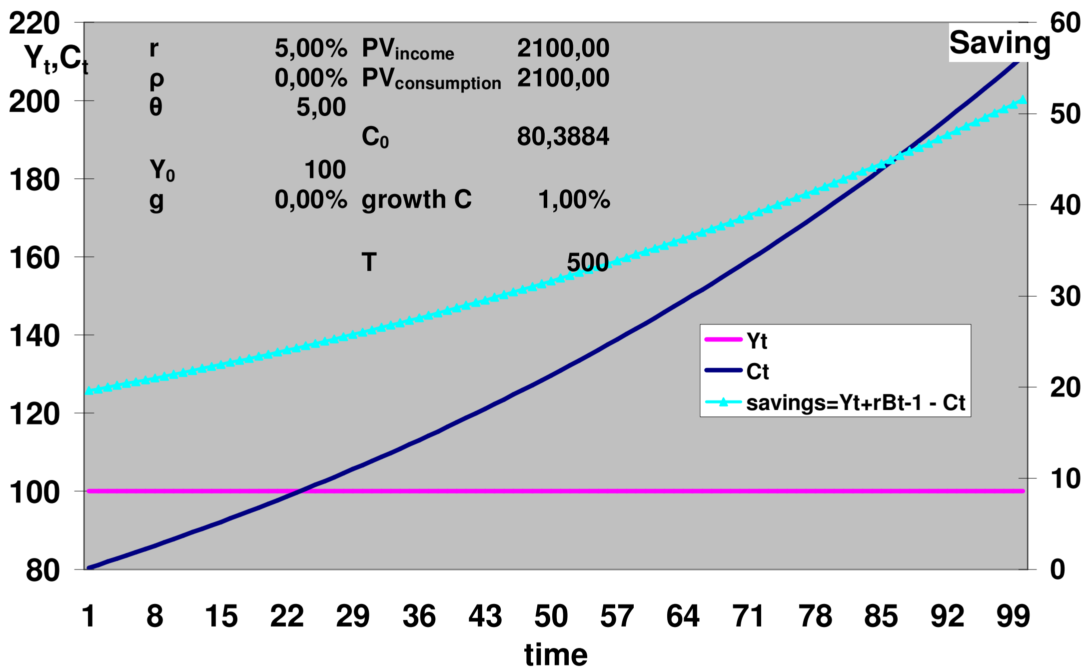


Figure no. 14_A4

Figures no.15_A4 and 16_A4 are designed for a higher elasticity of substitution (θ=2 and θ=1). As can be seen, with lower preference for consumption smoothing present optimal consumption is lower, because optimum consumption growth is higher. But it never declines to zero. Thus, even for the case of constant labour income, which is closest to the idea of an evenly rotating economy envisioned by Mises, zero time preference and positive real interest rate do not lead to a complete reduction of present consumption and postponement of all consumption to the end of the planning horizon. Even though this tendency might be deduced from lower θ and an extending planning horizon, Misesian story does not hold apart from a perfect-substitute case, which is characterised by a non-decreasing marginal utility of consumption (see Figure no. 17_A4).

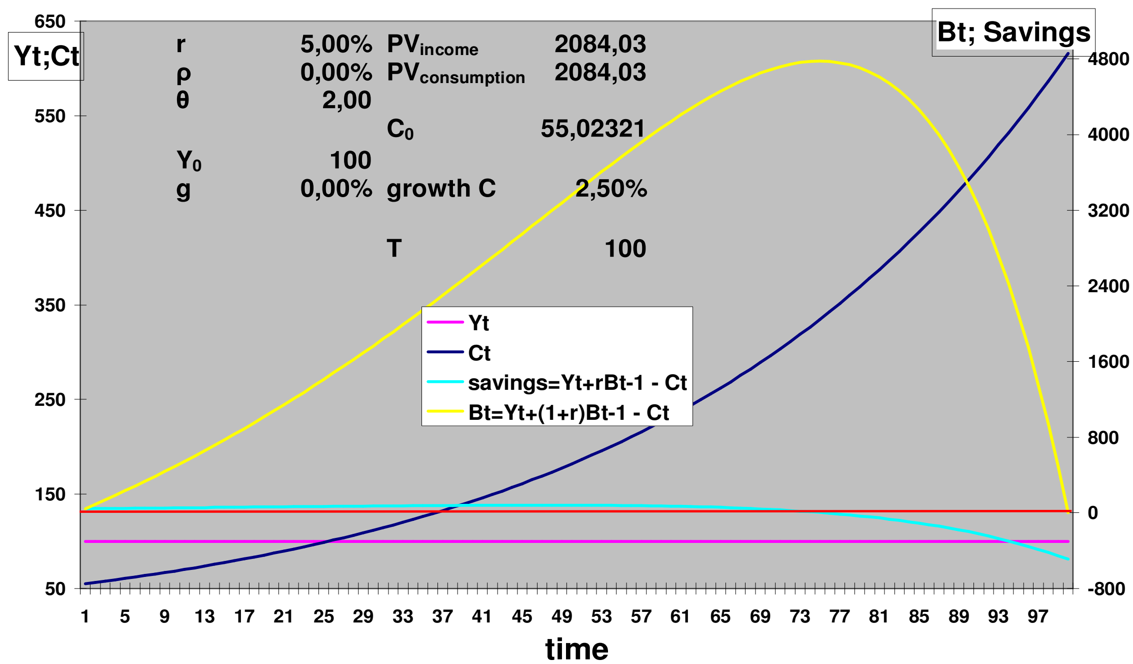


Figure no. 15_A4

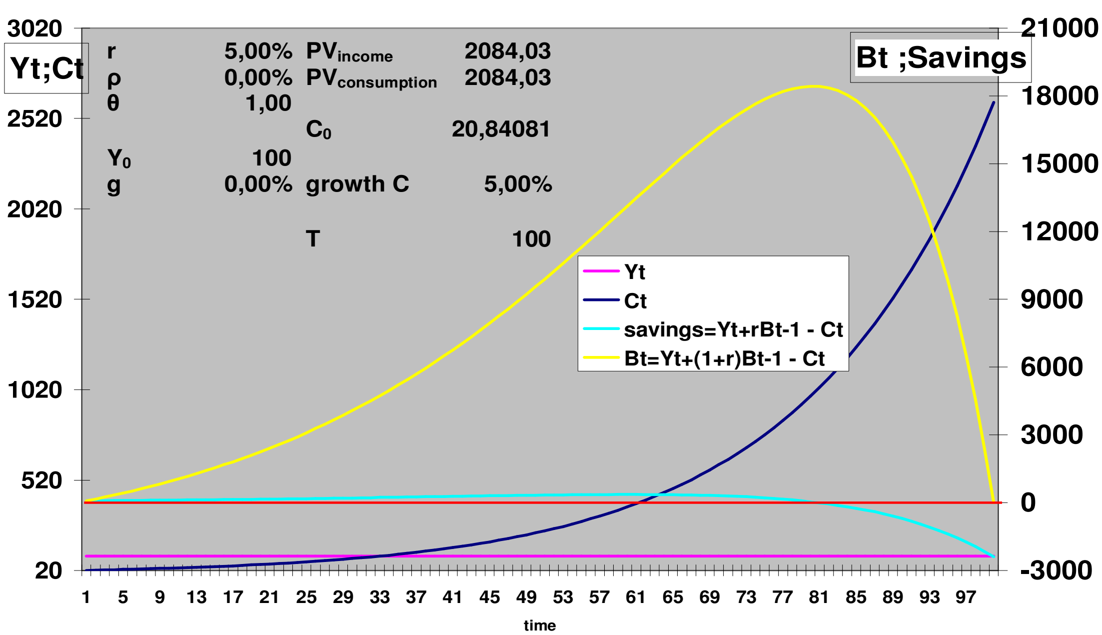


Figure no. 16_A4

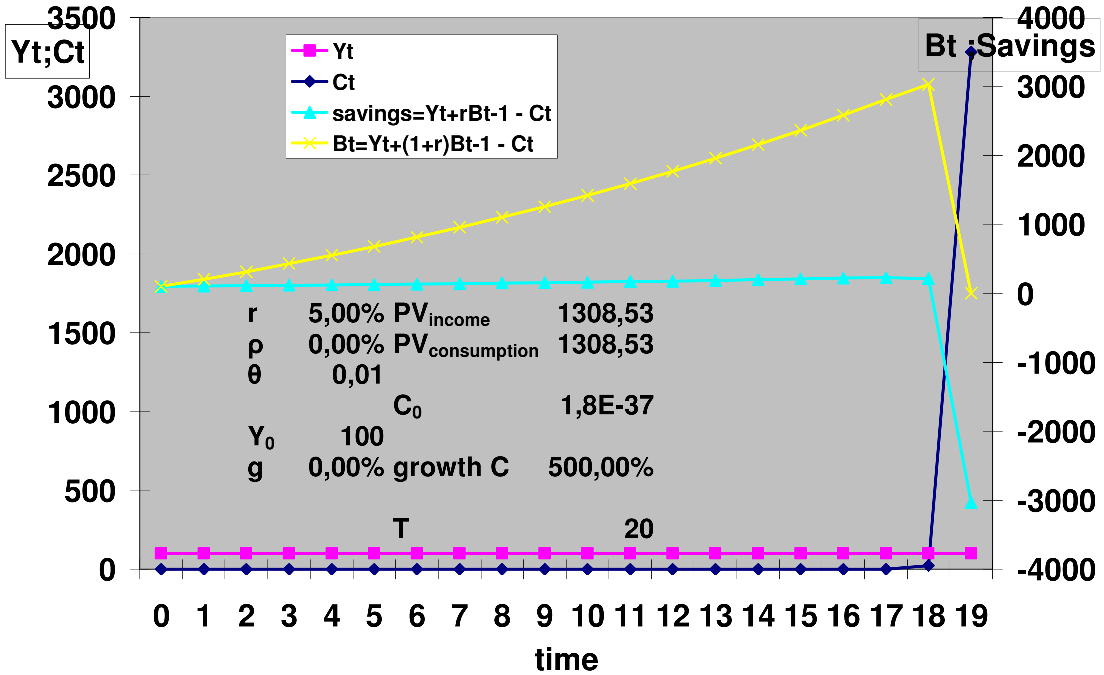


Figure no. 17_A4

It can be said that Mises underestimated the power of the law of diminishing marginal utility in his intertemporal analysis. This law results in an effort to smooth consumption over time, to provide enough consumption goods in every period. Thus, no period can be oversupplied with consumption goods at the expense of other periods even if the time preference is zero and the real interest rate is positive.

Only very long planning horizons and a moderately diminishing MU (i.e. low θ) suggest that present consumption might be depressed to zero. Thus, let us examine an infinite horizon model whether it gives credit to the Misesian reasoning. At the same time, we will explore the key predictions of Olson and Bailey about the necessity of drastically diminishing MU and an increasing profile of the income stream.

Infinite Horizon

The relevant intertemporal budget constraint for the infinite horizon is (A5_12) or (A5_13) in Appendix 5. Let us assume a constant real interest rate over time so as to easily compare our results with those of Olson and Bailey. By substituting the optimal consumption flow represented by (A4_9) into the intertemporal budget constraint (A5_13), we may write:

$$C_0 + \frac{\left(\frac{1+r}{1+\rho}\right)^{1/\theta} C_0}{(1+r)} + \frac{\left(\frac{1+r}{1+\rho}\right)^{2/\theta} C_0}{(1+r)^2} + \ldots = Y_0 + \frac{Y_1}{(1+r)} + \frac{Y_2}{(1+r)^2} + \ldots \qquad \text{(A4\_28)}$$

The left hand side of the intertemporal budget constraint (A4_28) might be written as:

$$PV_{consumption} = C_0 + C_0 \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} + C_0 \left[\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]^2 + ... \quad \text{(A4\_29)}$$

According to the formula of the sum of the infinite geometric series, (A4_29) gives us:

$$PV_{consumption} = \frac{1}{1 - \frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}} C_0 \quad \text{(A4\_30)}$$

$$PV_{consumption} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}} C_0 \quad \text{(A4\_31)}$$

However, the sum of this infinite series is finite only if:

$$(1+r)^{(1-\theta)/\theta} < (1+\rho)^{1/\theta} \quad \text{(A4\_32)}$$

$$\frac{1-\theta}{\theta} \ln(1+r) < \frac{1}{\theta} \ln(1+\rho) \quad \text{(A4\_33)}$$

If r and ρ are small numbers, (A4_33) yields:

$$1-\theta < \frac{\rho}{r} \quad \text{(A4\_34)}$$

$$\theta > 1 - \frac{\rho}{r} \quad \text{(A4\_35)}$$

In (A4_10) and the equations that ensued, we set θ=1 and ρ=0. We concluded that in such a case the optimal present consumption (and all future consumption levels except for infinity) was zero. Now, we can see that this set of parameters is not consistent even with the convergence of the sum of the flow of optimum consumption in the infinite horizon.[41] Zero time preference (ρ=0) is consistent only with θ>1, i.e. only with the utility function that exhibits relatively low elasticity of substitution. Alternatively we may say that such a utility function is characterized by a "dramatically diminishing marginal utility". Values of θ lower or equal to one would lead not only to the divergence of the sum of the flow of optimum consumption, but also to negative or zero optimum present consumption, as can be seen in equation (A4_37) below if we substitute ρ=0, θ≤1 and any positive real interest rate.

Furthermore, equation (A4_35) also implies that the higher the subjective discount rate (i.e. time preference in sense two), the higher the elasticity of substitution (lower θ) is feasible to achieve a convergent sum of the flow of consumption and positive optimum present consumption.

Using the formula for the present value of income, provided that the income process obeys $Y_t=(1+g)Y_{t-1}$, (A4_28) might be written as (see A5_21 in Appendix 5 for the PV of income) :

$$C_0 \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}} = Y_0 \frac{1+r}{r-g} \quad \text{(A4\_36)}$$

[41] Notice that the same result is obtained for the condition of the convergence of life-time utility. See section B in Appendix 5.

$$C_0 = Y_0 \; \frac{1+r}{r-g} \frac{(1+\rho)^{1/\theta} - (1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}} \qquad \text{(A4\_37)}$$

Using (A4_37) and (A4_9), the level of optimum consumption in any period might be then determined. Olson and Bailey suggested that if the time preference is zero (ρ=0), the Euler equation (A4_9) in the infinite horizon implies that present consumption is depressed to zero, unless the income endowment grows over time (i.e. g>0%) and the utility function exhibits a dramatically decreasing marginal utility (θ>1 in our framework). Our analysis confirms their second observation, θ>1 must be positive to obtain positive C_0. However, according to (A4_37), positive g might not be required.

Let us first explore the optimal level of present consumption, if income grows over time. In the second part, we will keep the income endowment constant. Both Mises (???) and Olson, Bailey (???) would predict that present consumption must be necessarily zero and every unit of disposable income must be saved and postponed to indefinite future provided that the time preference is zero. Equation (36) from the main text (here A4_38) seems to be in accordance with this conclusion:

$$\lim_{T\to\infty} \frac{C_0}{C_T} = \lim_{T\to\infty} \frac{1}{(1+r)^{T/\theta}} = 0 \qquad \text{(A4\_38)}$$

Olson and Bailey tried to escape from this result by assuming a positive growth in (labour) income endowment, which, according to them, can only guarantee infinite C in infinity and non-zero present consumption.

Again, let us run several simulations, this time for an infinite horizon. Olson and Bailey suggested that r = 5 %, g = 1 %, θ = 5 and ρ=0. These parameters lead to the growth rate in consumption of 1 % (see equation 37 in the main text), i.e. to the same growth rate as in the case of income. However, equation (37) is just an approximation. The correct growth rate is 0.98 % for this set of parameters (see 34 in the main text or A4_44 below), which is a little bit less than the growth rate in income. That is the reason why in Figures 1_A4 to 4_A4 we observed that present consumption exceeded present income and that saving was negative in the first part of the planning horizon and positive in the second part. These dynamics then resulted in a typical U-shaped behaviour of total debt B_t. Debt was first accumulated and then it was being paid off.

Nonetheless, the key intention of Olson and Bailey was surely to perfectly equalize the growth rate in income and consumption. Then they could argue that future consumption might be infinite even for a positive present consumption. In the infinite horizon, we must select parameters that will perfectly result in a 1 % growth rate in consumption otherwise the error from the approximation of equation (34) by equation (37) from the main text will expand beyond all limits.

Thus, our analysis of the finite horizon model presented above, which included this error, just demonstrated the optimum behaviour of consumption, saving and debt, if the optimum growth rate of consumption (0.98 %) is lower than the given growth rate in income (1 %). The subsequent simulations then provided dynamics for the set of parameters resulting in the optimum growth rate in consumption that exceeded the growth rate in income (which was either 1 % or 0 %). We saw that in such a case, assets were accumulated first and then used to keep the optimum consumption growing independently of the sluggish growth (or stagnation) in the labour income.

However, if r = 5 % and ρ = 0 %, θ that is required to generate a 1 % growth rate in consumption is 4.9 (see the procedure below from equation (A4_39) onwards), not 5 as suggested by Olson and Bailey.[42]

$$\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} = (1+g) \qquad \text{(A4\_39)}$$

$$\theta \ln(1+g) = \ln\left(\frac{1+r}{1+\rho}\right) \qquad \text{(A4\_40)}$$

$$\theta = \frac{\ln(1+r) - \ln(1+\rho)}{\ln(1+g)} \qquad \text{(A4\_41)}$$

Alternatively, if we want to determine the interest rate that would lead to the growth of consumption g for some given θ, we can modify (A4_39) to:

$$\frac{1+r}{1+\rho} = (1+g)^{\theta} \qquad \text{(A4\_42)}$$

$$r = (1+g)^{\theta}(1+\rho) - 1 \qquad \text{(A4\_43)}$$

If r, g and ρ are small numbers, (A4_40) might be written as:

$$r = \rho + \theta g \qquad \text{(A4\_43b)}$$

Thus, for r = 5 %, g = 1 %, θ = 4.9, ρ=0 (and zero initial assets) the model (both infinite and finite) predicts that optimum consumption perfectly coincides with income in every period. To prove this conclusion, let us notice that equation (A4_8) suggests that the optimum growth rate of consumption x is:

$$x = \frac{C_{t+1}}{C_t} - 1 = \left(\frac{1+r}{1+\rho}\right)^{1/\theta} - 1 \qquad \text{(A4\_44)}$$

(A4_30) then implies that:

$$PV_{consumption} = \frac{1}{1 - \frac{1+x}{1+r}} C_0 \qquad \text{(A4\_45)}$$

Because the PV of income must be equal to the PV of consumption, (A4_45) yields:

$$PV_{income} = PV_{consumption} = \frac{1+r}{r-x} C_0 \qquad \text{(A4\_46)}$$

By substituting the formula for the present value of income (A5_21), (A4_46) might be written as:[43]

[42] It should be stressed that we do not blame these authors for this approximation, because this set of parameters will perfectly hold in a continuous model (see equation ??? in section ??? in the main text). However, the infinite horizon model requires perfect accuracy, because any rounding error will be magnified sky high. On the other hand, this rounding error allowed us to report a specific behaviour of consumption and debt in the finite horizon model, which was useful as well.

$$\frac{1+r}{r-x}C_0 = Y_0 \frac{1+r}{r-g} \qquad \text{(A4\_47)}^{44}$$

$$C_0^* = Y_0 \frac{r-x^*}{r-g} \qquad \text{(A4\_48)}$$

As can be seen, the optimum present consumption is below present income, if the optimum growth rate of consumption is greater than the exogenous growth rate in income ($x^* > g$). In this case, the consumer is a saver. On the other hand, it is optimal to be a borrower, if the opposite assumption holds. The set of parameters leading to $x^* > g$ or $x^* < g$ can be found in (A4_71) and (A4_74) respectively, which is, however, derived from (A4_43). To take one example, $x^* > g$, i.e. the consumer is a lender, if the interest rate is large enough compared with the subjective discount rate and the growth rate in labour income. Furthermore, the lower the elasticity of substitution (higher θ), the higher interest rate is required to reach the lender position.

(A4_48) clearly demonstrates that the optimum present consumption is equal to the present income (C_0=Y_0=100), if $x^* = g$. It then grows at the rate of the labour income (g =1 %) reaching infinity in the infinite horizon. Saving is zero every period, so is the debt, hence the No-Ponzi-Game condition is satisfied (see A5_15 in Appendix 5). Hence, present consumption is not depressed to zero in an infinite horizon even in the absence of positive time preference and the presence of positive real interest rate At the same time, future consumption grows beyond all limits confirming the Euler equation (36) in the main text or (A4_38) in this Appendix. Consumption need not be postponed to indefinite future, since the positive growth rate in income will guarantee an infinite future consumption in infinity. Olson and Bailey were perfectly right in this respect, unintentionally refusing the theory of Ludwig von Mises.

The above discussion indicates that the solution of Olson and Bailey leads not only to zero present value of (future) assets (or debts) as time goes to infinity, i.e.:

$$\lim_{T\to\infty} \frac{B_T}{(1+\mathrm{r})^{\mathrm{T}}} = 0 \qquad \text{(A4\_49)}$$

but also to zero future value of (future) assets (or debts) as time grows beyond all limits, because no debt is ever issued or no asset is ever accumulated:

$$\lim_{T\to\infty} B_T = 0 \qquad \text{(A4\_50)}$$

However, condition (A4_50) is more restrictive than condition (A4_49). Moreover, the intertemporal budget constraint used here and presented in Olson, Bailey is consistent with (A4_49). Discussion in Appendix 5 suggests that the presence of the infinite discounting in (A4_49) is consistent not only with non-zero debt or assets, but even with an infinite debt or assets provided that the numerator grows more slowly than the denominator. Thus, (A4_50) might be non-zero even if (A4_49) is zero. As a result, much wider range of possibilities could be consistent with the Euler equation (A4_38), which requires either zero present consumption or an infinite future consumption, and the intertemporal budget constraint

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[43] From now on, we may indicate utility maximizing values by asterisk.

[44] Notice that for parameters r=5 %, g = 1 % and Y_0=100, the PV is 2625, which is the same figure as the PV for a 500-year planning horizon presented above. Thus, it seems at first glance that very long horizons might be approximated by an infinite horizon model.

(A4_2) (or alternatively with the No-Ponzi-Game condition A4_49). However, such a proof demands a very detailed analysis of the behaviour of B_t in the infinite horizon.

Before we explore the dynamics of debt (assets) in the infinite horizon, let us present a simple path of consumption that is consistent with non-zero present consumption, infinite future consumption (see A4_38) and with the intertemporal budget constraint (A5_13) (or A4_2 in the infinite horizon). This path also builds on the ideas developed in the finite horizon models presented above. Suppose that a representative agent has a constant flow of income for the entire infinite planning horizon (g = 0 %) and that the interest rate is positive (r>0). Suppose further that he reduces his present consumption only by a tiny amount below the level of his present income. Let us assume that in subsequent periods, his consumption will be the same as the given period labour income. Even with no additional restriction on further consumption, the initial saving enables him to reach infinite consumption in the future due to the infinite compounding of interest. A tiny amount of savings in the present results in the infinite value of assets in infinity provided that the interest rate is positive.

Moreover, our agent can consume even more than his given period income ($C_t > Y_t$) provided that the base of his assets is not undermined. The reason lies in the fact that in the infinite horizon, any positive value of assets will grow to infinite value, even though part of these assets is consumed in some future periods. Thus, the representative agent can consume even more than is the value of the disposable income in the given period ($C_t > Y_t + r_t B_{t-1}$), if the value of his assets is being kept above zero. It should be stressed that such a strategy is required due to condition (A4_49), because assets must grow more slowly than the interest rate to satisfy the intertemporal budget constraint (NPG with equality).

We do not claim that such a strategy will maximize life-time utility (A4_1) in the infinite horizon. However, it clearly demonstrates that there might exist a path of consumption that is consistent with zero time preference, positive real interest rate, positive (i.e. non-zero) present consumption, infinite future consumption, constant flow of income and also with the intertemporal budget constraint.

We already know that the utility maximizing path of consumption must obey the Euler equation. Equation (A4_44) then determines the optimum growth rate of consumption x. If this growth rate is lower than the interest rate, the present value of consumption in infinity might be zero, even if the size of consumption as such is infinite and equation (A4_38) is thus satisfied.

To find the set of parameters that is consistent with this condition, let us realize that the present value of optimum consumption in infinity is (using A4_9):

$$\lim_{T\to\infty}\frac{C_T}{(1+r)^T}=\lim_{T\to\infty}\frac{C_0\left(\frac{1+r}{1+\rho}\right)^{T/\theta}}{(1+r)^T} \qquad \text{(A4\_51)}$$

Zero present value requires:

$$\left(\frac{1+r}{1+\rho}\right)^{1/\theta}<(1+r) \qquad \text{(A4\_52)}$$

or

$$(1+r)^{(1-\theta)/\theta}<(1+\rho)^{\theta} \qquad \text{(A4\_53)}$$

However, (A4_53) leads to condition (A4_35) thoroughly discussed before. Assuming zero time preference (ρ=0%), the present value of future consumption might be zero (see A4_51) in the infinite horizon even if the future value is infinite (the numerator in A4_51) only for θ>1. Surprisingly, we again arrived at our well known condition of dramatically decreasing marginal utility. Notice that we have not assumed anything about the specific time shape of the labour income.

Thus, let us here use both ideas from previous paragraphs. There might exist a consumption path with non-zero present consumption and infinite future consumption that is both consistent with the intertemporal budget constraint (NPG condition) and with the Euler equation, notwithstanding the time shape of the labour income stream.

Let us now explore the time path of debt (or assets), if the optimum path of consumption is chosen in the infinite horizon. We already know the optimum growth rate of consumption x^* (see A4_44) and the optimum level of present consumption C_0^* (A4_48). The key question is whether zero time preference requires both an increasing time shape of labour income and dramatically diminishing marginal utility, or if some of these assumptions might be omitted. Our goal is also to determine the resulting development of debt (or assets) in such a case.

The value of optimal assets (or debt) at time t is (see Appendix 5 for the general law of motion of B_t):

$$B_t^* = Y_t + B_{t-1}^*(1+r) - C_t^* \qquad \text{(A4\_54)}$$

Alternatively, (A4_54) might be written as:

$$B_t^* = Y_0 (1+g)^t + B_{t-1}^*(1+r) - C_0^*(1+x^*)^t \qquad \text{(A4\_55)}$$

Let us solve this difference equation iteratively. The optimum size of assets (or debt) in the present is:

$$B_0^* = Y_0 - C_0^* \qquad \text{(A4\_56)}$$

One period ahead, we may write:

$$B_1^* = Y_1 + B_0^*(1+r) - C_1^* \qquad \text{(A4\_57)}$$

This leads to:

$$B_1^* = B_0^*(1+r) + Y_0 (1+g) - C_0^*(1+x^*) \qquad \text{(A4\_58)}$$

In period 2, (A4_55) is:

$$B_2^* = Y_2 + B_1^*(1+r) - C_2^* \qquad \text{(A4\_59)}$$

By plugging (A4_57):

$$B_2^* = (1+r)\left[B_0^*(1+r) + Y_0 (1+g) - C_0^*(1+x^*)\right] + Y_0 (1+g)^2 - C_0^*(1+x^*)^2 \qquad \text{(A4\_60)}$$

And by inserting (A4_56) and rearranging terms:

$$B_2^* = (1+r)^2 Y_0 + (1+r)(1+g)Y_0 + Y_0 (1+g)^2 - (1+r)^2 C_0^* - (1+r)(1+x^*)C_0^* - C_0^*(1+x^*)^2$$

(A4_61)

At the end of the finite planning horizon T, (A4_61) can be generalized to:

$$B_T^* = (1+r)^T Y_0 + (1+g)(1+r)^{T-1} Y_0 + ... + (1+g)^T Y_0 - (1+r)^T C_0^* - (1+x^*)(1+r)^{T-1} C_0^* - ... - (1+x^*)^T C_0^* \qquad \text{(A4\_62)}$$

Adding up all terms with income and consumption separately, (A4_62) might be written as:

$$B_T^* = (1+r)^T Y_0 \frac{1-\left(\frac{1+g}{1+r}\right)^{T+1}}{1-\frac{1+g}{1+r}} - (1+r)^T C_0^* \frac{1-\left(\frac{1+x^*}{1+r}\right)^{T+1}}{1-\frac{1+x^*}{1+r}} \qquad \text{(A4\_63)}$$

$$B_T^* = Y_0 \frac{(1+r)^T - \frac{(1+g)^{T+1}}{1+r}}{\frac{1+r-1-g}{1+r}} - C_0^* \frac{(1+r)^T - \frac{(1+x^*)^{T+1}}{1+r}}{\frac{1+r-1-x^*}{1+r}} \qquad \text{(A4\_64)}$$

$$B_T^* = \frac{Y_0}{r-g}\left[(1+r)^{T+1} - (1+g)^{T+1}\right] - \frac{C_0^*}{r-x^*}\left[(1+r)^{T+1} - (1+x^*)^{T+1}\right] \qquad \text{(A4\_65)}$$

In the infinite time horizon, (A4_65) yields:

$$\lim_{T\to\infty} B_T^* = \lim_{T\to\infty}\left\{\frac{Y_0}{r-g}\left[(1+r)^{T+1} - (1+g)^{T+1}\right] - \frac{C_0^*}{r-x^*}\left[(1+r)^{T+1} - (1+x^*)^{T+1}\right]\right\} \qquad \text{(A4\_66)}$$

Using the expression for the optimum present consumption (A4_48), we may write:

$$\lim_{T\to\infty} B_T^* = \lim_{T\to\infty}\left\{\frac{Y_0}{r-g}\left[(1+r)^{T+1} - (1+g)^{T+1}\right] - \frac{Y_0 \frac{r-x^*}{r-g}}{r-x^*}\left[(1+r)^{T+1} - (1+x^*)^{T+1}\right]\right\} \qquad \text{(A4\_67)}$$

$$\lim_{T\to\infty} B_T^* = \lim_{T\to\infty}\left\{\frac{Y_0}{r-g}\left[(1+x^*)^{T+1} - (1+g)^{T+1}\right]\right\} \qquad \text{(A4\_68)}$$

As can be seen, the eventual value of debt (or assets) is zero only if the growth rate of optimum consumption is perfectly equal to the growth rate of labour income (x^*=g). The set of parameters leading to this outcome can be derived from (A4_43):

$$\ln(1+r) = \theta \ln(1+g) + \ln(1+\rho) \qquad \text{(A4\_69)}$$

If r, g and ρ are small numbers, the resulting interest rate is:

$$r = \rho + \theta g \qquad \text{(A4\_69b)}$$

However, if these growth rates differ, the total assets grow either to positive infinity or to negative infinity (so to infinite debt). First, if the growth rate of optimum consumption is greater than the growth rate in labour income (i.e. $x^*>g$), assets will be accumulated beyond all limits, even so the NPG condition will be satisfied as we will show below. We will also see that it is the initial accumulation of assets that will finance relatively greater growth rate of consumption to the infinite future. The formal proof for the statement above is as follows: Provided that $x^*>g$, (A4_68) might be written as follows:

$$\lim_{T\to\infty} B_T^* = \frac{Y_0}{r-g}\lim_{T\to\infty}(1+x^*)^{T+1}\left[1-\left(\frac{1+g}{1+x^*}\right)^{T+1}\right] = +\infty \qquad \text{(A4\_70)}$$

The set of parameters leading to $x^*>g$ can be derived from (A4_39) and (A4_43):

$$r > (1+g)^{\theta}(1+\rho)-1 \qquad \text{(A4\_71)}$$

If r, g and ρ are small numbers, (A4_71) yields:

$$r > \rho + \theta g \qquad \text{(A4\_72)}$$

Thus, the real interest rate must be relatively high to generate an infinite amount of assets in infinity. On the other hand, if the growth rate of optimum consumption is lower than the growth rate of the labour income (i.e. $x^*< g$), an infinite debt will be accumulated even though the rate of the debt accumulation must be lower than the interest rate to satisfy the NPG condition. The proof might use (A4_68) again:

$$\lim_{T\to\infty} B_T^* = \frac{Y_0}{r-g}\lim_{T\to\infty}(1+g)^{T+1}\left[\left(\frac{1+x^*}{1+g}\right)^{T+1}-1\right] = -\infty \qquad \textbf{(A4\_73)} \quad \text{(A4\_70)}$$

The set of parameters that will lead to such a result is:

$$r < (1+g)^{\theta}(1+\rho)-1 \qquad \text{(A4\_74)}$$

Or alternatively:

$$r < \rho + \theta g \qquad \text{(A4\_75)}$$

The real interest rate must be relatively low, people must be impatient enough or the labour income must grow rapidly along with a relatively low elasticity of substitution (low 1/θ). Then the debt will increase sky high in the infinite horizon.

Let us now examine the combination of parameters that will satisfy the NPG condition (A4_49). Using (A4_68), the NPG becomes:

$$\lim_{T\to\infty}\frac{B_T^*}{(1+r)^T} = \lim_{T\to\infty}\frac{\frac{Y_0}{r-g}\left[(1+x^*)^{T+1}-(1+g)^{T+1}\right]}{(1+r)^T} \qquad \textbf{(A4\_76)} \text{ (A4\_71)}$$

By substituting the optimum growth rate of consumption x* (see A4_44), we can write:

$$\lim_{T\to\infty}\frac{B_T^*}{(1+r)^T}=\frac{Y_0}{r-g}\lim_{T\to\infty}\frac{\left[\left(\frac{1+r}{1+\rho}\right)^{1/\theta}\right]^{T+1}-(1+g)^{T+1}}{(1+r)^T} \qquad \textbf{(A4\_77)}\ (A4\_72)$$

$$\lim_{T\to\infty}\frac{B_T^*}{(1+r)^T}=\frac{Y_0}{r-g}\lim_{T\to\infty}\frac{(1+r)^{-T+(T+1)/\theta}}{(1+\rho)^{(T+1)/\theta}}-\frac{Y_0}{r-g}\lim_{T\to\infty}(1+g)\left(\frac{1+g}{1+r}\right)^T \qquad \textbf{(A4\_78)}\ (A4\_73)$$

Because r is greater than g by assumption, the last part in the expression above is zero. (A4_78) might be the then written as:

$$\lim_{T\to\infty}\frac{B_T^*}{(1+r)^T}=\frac{Y_0}{r-g}\lim_{T\to\infty}\frac{(1+r)^{(T+1-T\theta)/\theta}}{(1+\rho)^{(T+1)/\theta}} \qquad \textbf{(A4\_79)}\ (A4\_74)$$

$$\lim_{T\to\infty}\frac{B_T^*}{(1+r)^T}=\frac{Y_0}{r-g}\left(\frac{1+r}{1+\rho}\right)^{1/\theta}\lim_{T\to\infty}\left(\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right)^T \qquad \textbf{(A4\_80)}\ (A4\_75)$$

(A4_80) converges to zero, i.e. the NPG condition is satisfied with equality if and only if:

$$(1+r)^{(1-\theta)/\theta}<(1+\rho)^{1/\theta} \qquad \textbf{(A4\_81)}\ (A4\_76)$$

But this condition was derived many times before (see A4_53, A4_32 and even A5_29). At this place we can conclude that zero time preference (ρ = 0 %) is consistent with NPG, only for a dramatically diminishing marginal utility (θ>1).

The solution of standard neoclassical growth model implies that on the balanced growth path (at the steady state) the growth rate of the optimum consumption equals the growth rate in labour income. This assumption was also employed by Olson and Bailey. The resulting equilibrium interest rate is then (A4_43). Substituting this equation into (A4_81), we can modify our requirement to:

$$\left[(1+g)^{\theta}(1+\rho)\right]^{(1-\theta)/\theta}<(1+\rho)^{1/\theta} \qquad \textbf{(A4\_82)}\qquad (A4\_77)$$

$$(1-\theta)\ln(1+g)+\frac{(1-\theta)}{\theta}\ln(1+\rho)<\frac{1}{\theta}\ln(1+\rho) \qquad \textbf{(A4\_83)}\qquad (A4\_78)$$

$$(1-\theta)\ln(1+g)-\ln(1+\rho)<0 \qquad \textbf{(A4\_84)}\ (A4\_79)$$

If g and ρ are small numbers, we can write:

$$\rho-(1-\theta)g>0 \qquad \textbf{(A4\_85)}\ (A4\_80)$$

On the balanced growth path (where the growth rates of consumption and income are equal), this condition guarantees stability in many parts of our model. If the time preference is zero, income growth must be positive and marginal utility must be dramatically diminishing (θ>1).

However, in our analysis we also explore possibilities of different growth rates of income and optimum consumption. (A4_81) is then the required condition. Before we simulate the behaviour of the key variables, let us examine the optimum rate of debt or asset accumulation in the infinite horizon. In the finite horizon, debt or assets must be zero at the end of the planning horizon (i.e. B_T=0). In the infinite horizon, this is true only if the growth rate of consumption and labour income coincide.

Nevertheless, if these growth rates differ, assets are either infinitely accumulated or debt is infinitely issued. We would like to find the optimum rate of this debt or assets expansion, that is, however, lower than the interest rate, because the NPG condition must be satisfied. Using (A5_16), the optimum growth rate of assets in the infinite horizon is:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{Y_T-C_T^*}{B_{T-1}^*}+r\right] \qquad \text{(A4\_86)}$$

By inserting (A4_68) for B^*_{T-1}:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{Y_0(1+g)^T-C_0^*(1+x^*)^T}{\frac{Y_0}{r-g}\left[(1+x^*)^T-(1+g)^T\right]}+r\right] \qquad \text{(A4\_87)}$$

If we substitute (A4_48) for the optimum present consumption:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{Y_0(1+g)^T-Y_0\,\frac{r-x^*}{r-g}(1+x^*)^T}{\frac{Y_0}{r-g}\left[(1+x^*)^T-(1+g)^T\right]}+r\right] \qquad \text{(A4\_88)}$$

A simple rearrangement of terms gives us:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{(r-g)(1+g)^T-(r-x^*)(1+x^*)^T}{\left[(1+x^*)^T-(1+g)^T\right]}+r\right] \qquad \text{(A4\_89)}$$

If the growth rate of optimum consumption is greater than the growth rate in labour income (i.e. x^*>g), (A4_89) might be written as:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{(r-g)\left(\frac{1+g}{1+x^*}\right)^T-(r-x^*)}{1-\left(\frac{1+g}{1+x^*}\right)^T}+r\right] \qquad \text{(A4\_90)}$$

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=-r+x^*+r=x^*=\left(\frac{1+r}{1+\rho}\right)^{1/\theta}-1 \qquad \text{(A4\_91)}$$

Condition (A4_81) guarantees that x^* is lower than the interest rate r. Thus, if the growth rate of optimum consumption is higher than the growth rate in labour income, the optimum assets grow beyond all limits (see A4_70) at the rate of $x^* < r$.

On the other hand, if the growth rate of optimum consumption is lower than the growth rate in labour income (i.e. $x^*<g$), (A4_89) can be rearranged as:

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=\lim_{T\to\infty}\left[\frac{(r-g)-(r-x^*)\left(\frac{1+x^*}{1+g}\right)^T}{\left(\frac{1+x^*}{1+g}\right)^T-1}+r\right] \qquad \text{(A4\_92)}$$

$$\lim_{T\to\infty}\frac{\Delta B_T^*}{B_{T-1}^*}=-(r-g)+r=g \qquad \text{(A4\_93)}$$

As can be seen, if the growth rate of optimum consumption is lower than the growth rate in labour income, the optimum debt grows beyond all limits (see A4_73) at the rate of $g < r$.

Now, we can utilize all the information we have just derived to run simple simulations in an infinite horizon. Our main objective is to test both the Misesian prediction about the reduction of present consumption to zero in the absence of time preference and the Olson and Bailey´s prediction that the income growth must be positive to obtain non-zero present consumption if the subjective discount rate is zero.

Let us start with the situation when the growth rate in labour income exceeds the optimum growth rate in consumption. This might be achieved, if we use the approximation (37) in the main text instead of the accurate equation (34). Suppose (like Olson and Bailey) that r = 5 %, ρ = 0 %, θ = 5 and g = 1 %. The precise optimum growth rate of consumption is 0.98%. This set of parameters is consistent with condition (A4_32).

Figures no. 18_A4 – 22_A4 show the resulting dynamics of consumption, income, total debt, saving, the growth rate of debt and the present value of debt. Various time horizons are reported for further clarity. First, the real interest rate (r = 5 %) is greater than the subjective discount rate (ρ = 0 %), so the optimum consumption grows over time (see equation A4_8). Second, this optimum growth rate of consumption (x^*=0.98 %) is lower than the growth rate in income (g = 1 %), thus equation (A4_48) implies that the present optimum consumption exceeds present income (see Figure no. 18_A4). As can be seen, present consumption is not depressed to zero, as would be predicted by L.von Mises.

Third, negative saving in the first part of the planning horizon (the consumer is a borrower) leads to an increasing debt over time. Even though the consumption gradually falls below the given period labour income, saving is still negative due to the repayment of interest from the increasing debt. As a result, debt is not being repaid and it grows (B_t is more and more negative).[45] In the infinite horizon, the NPG condition, along with the intertemporal budget constraint, only requires that the growth rate in debt is lower than the interest rate. In this particular case, debt reaches an infinite value in infinity (see equation A4_73) growing at the rate of g (see A4_93), even though its present value gradually approaches zero. The optimum growth rate of consumption falls short of the income growth due to the eternal payment of interest. The economic intuition behind this behaviour is as follows: People, in the expectation of higher future income, increase their present consumption above the present period income. This choice is reinforced by relatively high θ, which motivates the individual to smooth consumption over time. High future consumption levels attained due to rapid increase in labour income are moved closer to present owing to negative saving and the accumulation of debt.

Thus, Olson and Bailey´s prediction seems to be correct. Zero time preference does not lead to zero present consumption. Its level might even exceed the present labour income, if the preference for consumption smoothing is really high resulting in a low optimum growth rate in consumption.

Figures no. 23_A4 – 25_A4 display optimum paths for θ=2. As we can see, lower preference for consumption smoothing (less dramatically diminishing marginal utility), results in a higher growth in optimum consumption (x^*=2.47 %), which in our case exceeds the growth rate in labour income (g=1 %). Present consumption is thus depressed below present labour income (the consumer is a lender, see A4_48) and assets are gradually accumulated. This asset accumulation is eternal fuelling higher consumption growth. As can be seen, even though the labour income will fall below consumption in the future, the interest income earned from the previous saving allows consumption to grow faster than the labour income. Thus, consumption in very remote future seems to be quite independent of that period labour income. Furthermore, notice that saving need not be especially large in the present and in early periods to generate enough assets. Hence, present consumption is not depressed to zero. Eternal interest earned on assets and the growth of consumption that is lower than the interest rate ($x^* < r$) result in the fact that assets grow forever (see equation A4_70). The growth rate at which the assets are growing in the infinite horizon is x^* (see A4_91).

The previous analysis suggests that the accumulation of interest might suffice to feed high consumption growth regardless of the behaviour of the labour income. Let us suppose that the labour income is constant. This assumption is closest to the Misesian ERE. Other assumptions from the previous sections will be kept. Thus, the real interest rate is positive, r = 5 % and the time preference (in sense two) is zero. Mises would suggest that the present consumption must be depressed to zero and the act of consumption will never occur. Olson and Bailey require a positive income growth for the parameters above. If it is constant, they predict the same outcome as L. von Mises.

However, let us demonstrate that present consumption might be positive even with constant labour income stream. The only requirement is a relatively low elasticity of substitution (θ>1),

[45] Notice that even though the PV is (almost) the same as in the 500-year finite horizon model and the present optimum consumption is also very close, debt is not repaid in period 500. The reason is that in the infinite horizon, the present consumption is a little bit higher so is the consumption in every subsequent period.

i.e. what Olson and Bailey called a dramatically diminishing marginal utility. Figures no. 26_A4 – 28_A4 display the development of the key variables for $\theta=5$. As can be seen, optimum consumption growth is positive ($x^*=0.98$ % again), because the real interest rate exceeds the subjective discount rate. Present consumption is lower than present income, which leads to positive present saving. The same observation holds for a few early periods. As a result, positive assets are accumulated, which allows consumption to exceed labour income in the future. Compared with the situation of positive income growth (g=1%), present consumption cannot be greater than present labour income. Nevertheless, it is not depressed to zero as Mises and Olson and Bailey would suggest. In the infinite horizon, the initial accumulation of assets generates enough capital income in the future to fuel eternal consumption growth. At the same time, assets are growing at the rate of x^*, because $x^*= 0.98$ % is greater than $g = 0$ % (see the discussion from A4_86 to A4_91), reaching positive infinity in the infinite horizon (see A4_70). Notice, that the NPG condition (the intertemporal budget constraint) is satisfied with equality, because the present value of assets gradually falls to zero (Figure no. 27_A4).

The economic reason for such behaviour is as follows. If the marginal utility is sufficiently decreasing, the additional units of consumption goods satisfy wants of much lower urgency. Even for zero time preference (in sense two), i.e. if the given want is not preferred to be satisfied as soon as possible, and for the positive interest rate, the representative consumer has a tendency to spread consumption goods evenly over time. Overprovision of goods in the future and starving in the present cannot be optimal, because present wants of very high urgency would stay ungratified, whereas the abundance of future consumption goods will satisfy wants of only very low importance. As a result, the optimum behaviour is not to reduce present consumption to zero, but to some reasonable levels. On the other hand, these levels must be sufficiently low to accumulate enough assets in the early periods. Then, the consumer may enjoy future interest income as a source for eternal consumption growth. Thus, there is no need for the labour income to grow over time to satisfy Olson and Bailey´s condition (36) from the main text. The eternal accumulation of interest will guarantee infinite future consumption and the present consumption need not fall to negligible levels.

Our early analysis suggested that as θ approaches 1, the present consumption is depressed to zero provided that the time preference does not exist ($\rho = 0$ %). We also demonstrated that if $\rho = 0$ % the stability of the model requires $\theta>1$. Figures no. 29_A4 – 31_A4 clearly show that if the utility function approaches the logarithmic form ($\theta \to 1$), the solution of Mises is much more plausible to emerge. Consumption in the present and in the early periods is negligible for the benefit of the future. Assets are growing at the rate of $x^*=4.95$ %. This figure is very close to the size of the interest rate. As a result, the present value of assets converges to zero at a very slow pace (see Figure no. 30_A4).

We can conclude that the Misesian theory holds only under special circumstances characterised by a relatively high elasticity of intertemporal substitution in consumption ($\theta \leq 1$), i.e. the utility function must exhibit weakly diminishing marginal utility. In such a case, the absence of time preference leads to zero consumption in all times except for infinity. Thus, Olson and Bailey suggested that the utility function must exhibit "drastically diminishing" marginal utility and income must grow over time. However, we clearly demonstrated that the path of labour income is quite immaterial, provided that its growth rate is lower than the interest rate.

Let us even consider a diminishing time shape of the labour income stream. Even in this case, the requirement (A4_53) guarantees an infinite future consumption and positive and relatively high (depending on θ) present consumption. See Figures no. 32_A4 – 34_A4 that report behaviour of the key variables for $r = 5$ %, $\theta = 5$, $\rho = 0$ % and $g = -1$ %. As can be seen, they are almost indistinguishable from Figures no.26_A4 - 28_A4. The only difference is lower present consumption that is required to accumulate enough assets serving as a source for gradually increasing future consumption.

As an extreme case of diminishing labour income endowment, we may consider a Fisherian infinitely lived sailor shipwrecked with a stock of goods that have a constant productive power (e.g. herd of sheep or rice). Suppose that the initial endowment of this good is 100. It cannot be enlarged, i.e. the future income endowment is 0 (g = -100 %), unless it is wisely invested. The marginal productivity of this good is constant MPK = r = 5 %. Suppose that the subjective discount rate is 0 %, and θ = 5. As can be seen in Figures no. 35_A4 – 37_A4, the sailor might enjoy an increasing consumption flow (x*= 0.98 %) reaching infinity in the infinite horizon, even though his labour income is zero over the entire lifetime (formally except for the present). All sources of future consumption consist of capital income that is generated owing to considerable saving in early periods. Although this saving is relatively high, present consumption is not depressed to zero even in this extreme case.

At the end of this Appendix, let us stress that our analysis examined the optimal behaviour of one representative agent. We tried to demonstrate that the theory of L. von Mises(???) and Olson and Bailey (???) are rather inaccurate. However, the solution offered by the later, i.e. the uniform growth rate of income and consumption, seems to be sensible for the aggregate economy. Our case of constant labour income stream and increasing path of consumption led to an eternal accumulation of assets. It is quite difficult to generalize such behaviour to the entire economy. It seems to be quite reasonable to assume that the surplus of saving from the beginning of the planning horizon will depress the interest rate to the level of the subjective discount rate, i.e. to zero. However, this would violate condition (A4_53) and the model would collapse. Thus, zero saving, zero eternal debt (assets), positive interest rate and zero time preference seem to be a stable situation at the aggregate level only for an increasing income stream and low elasticity of substitution (θ>1).

SIMULATIONS 2

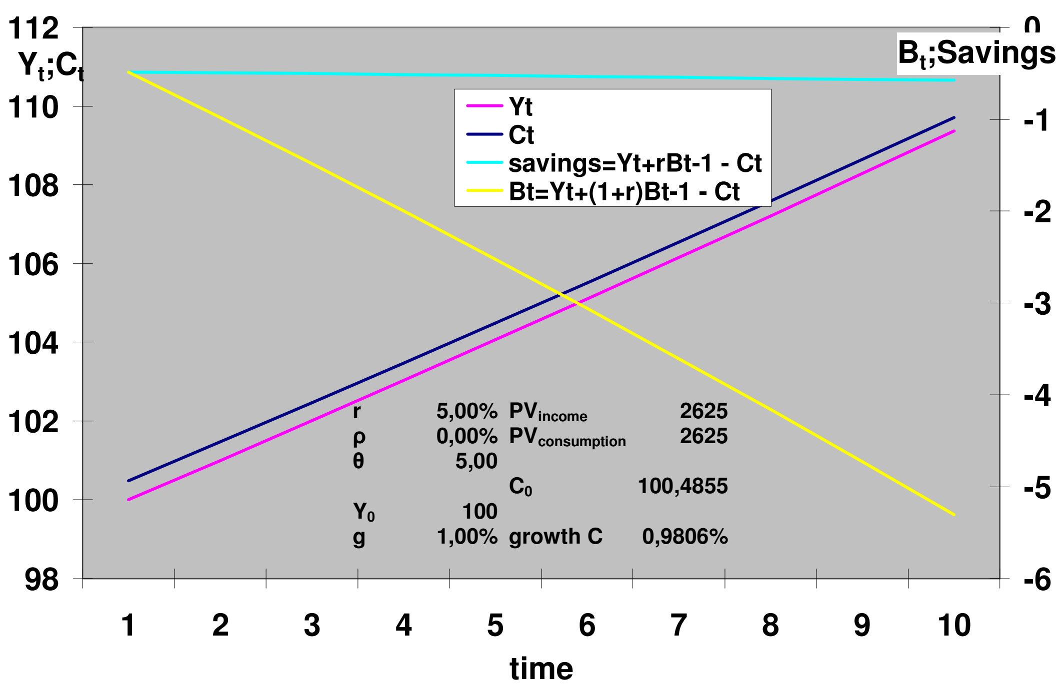
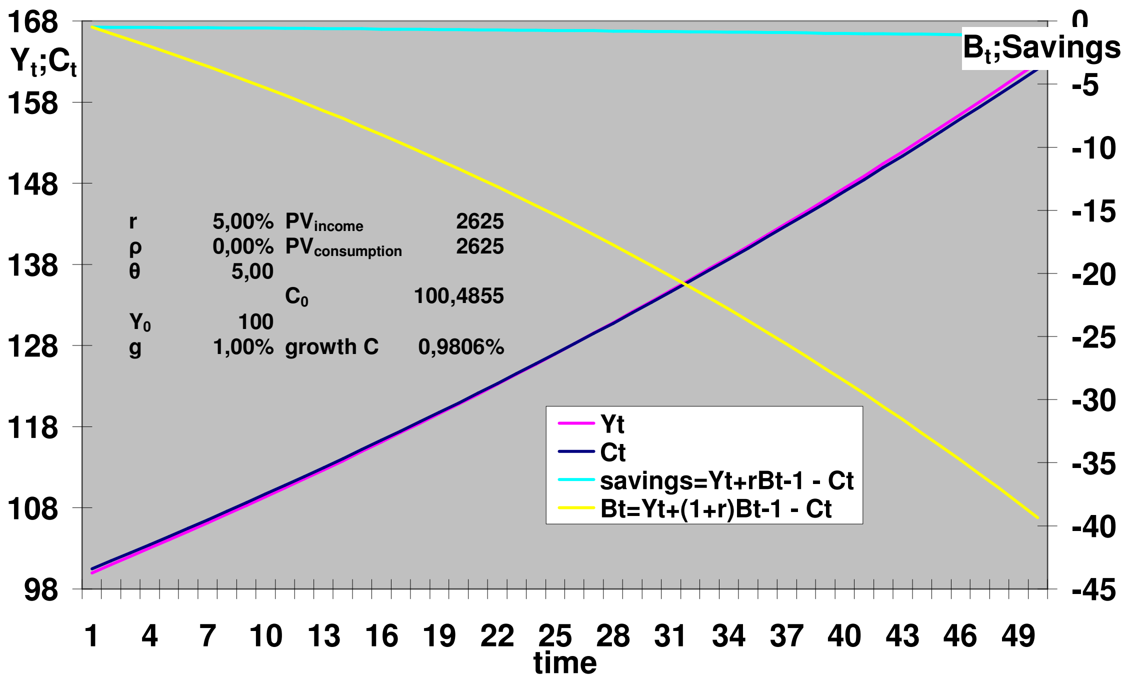


Figure no. 18_A4



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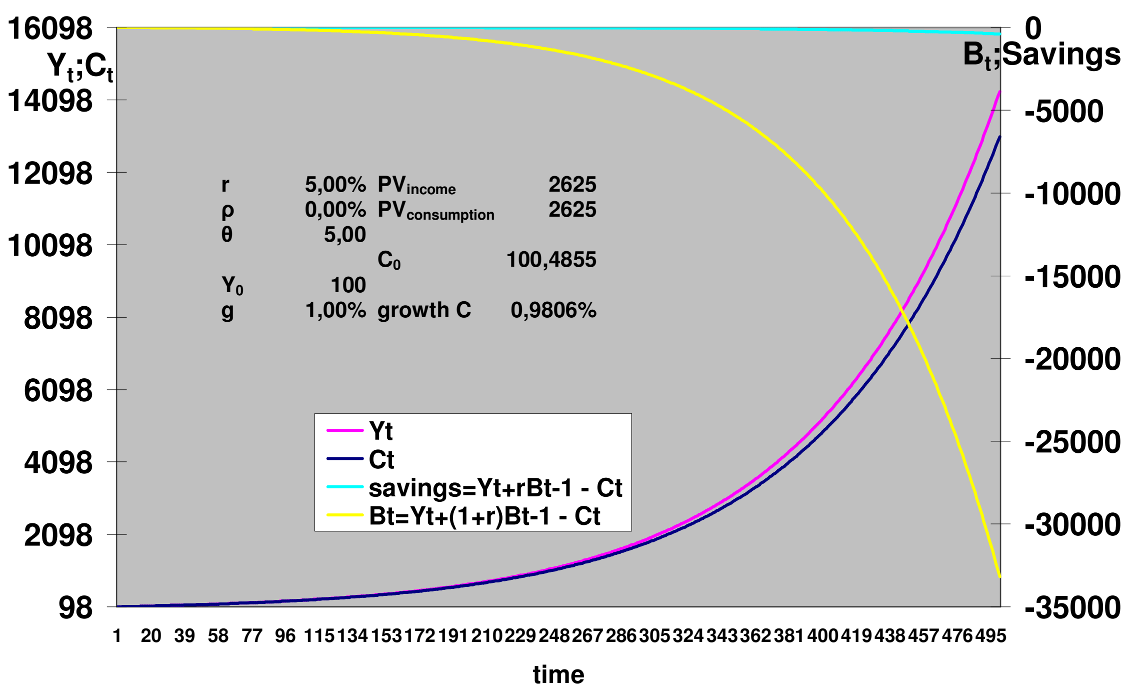

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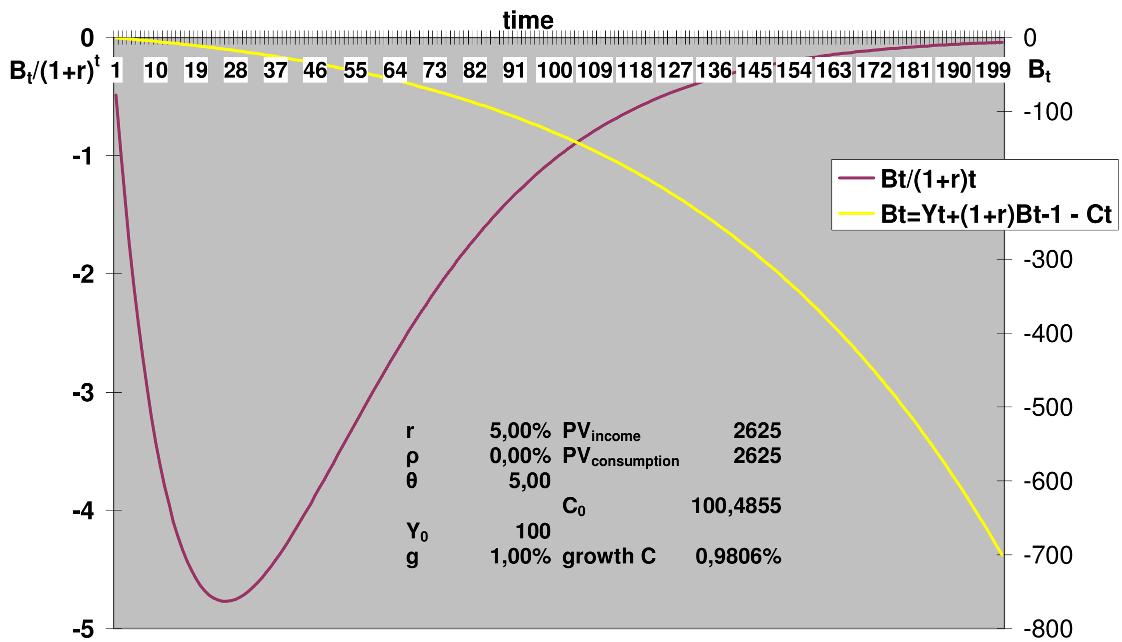

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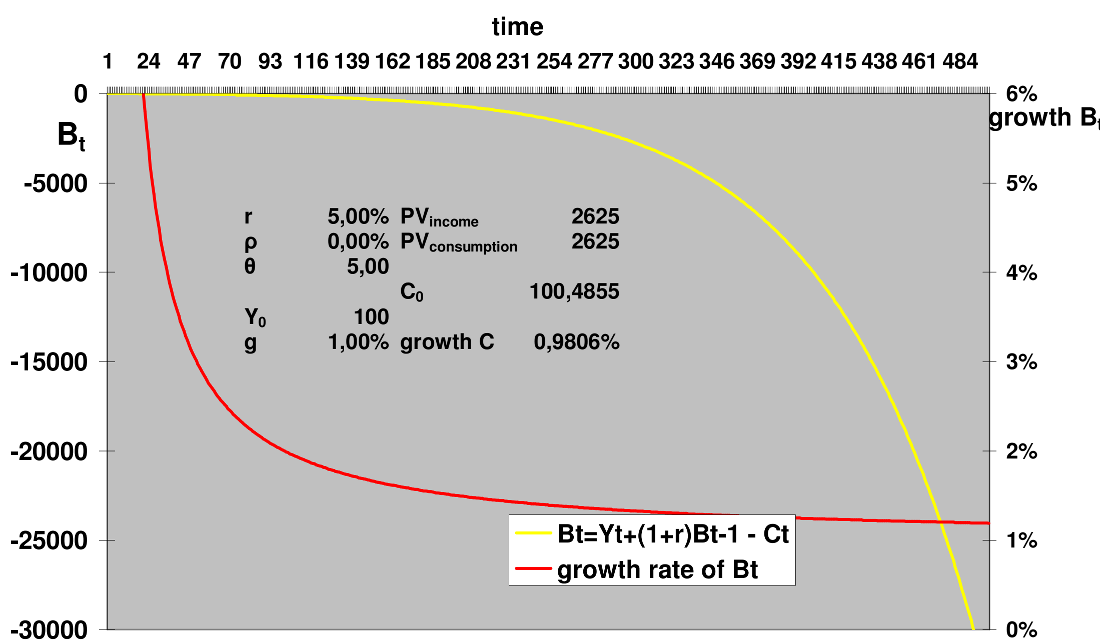

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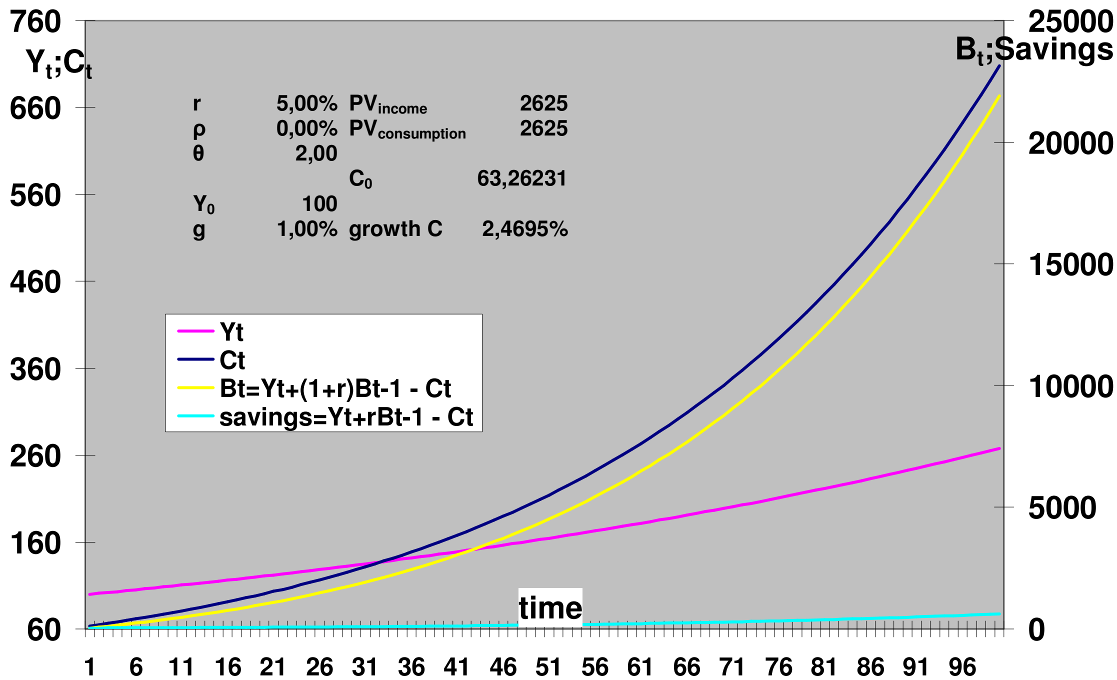

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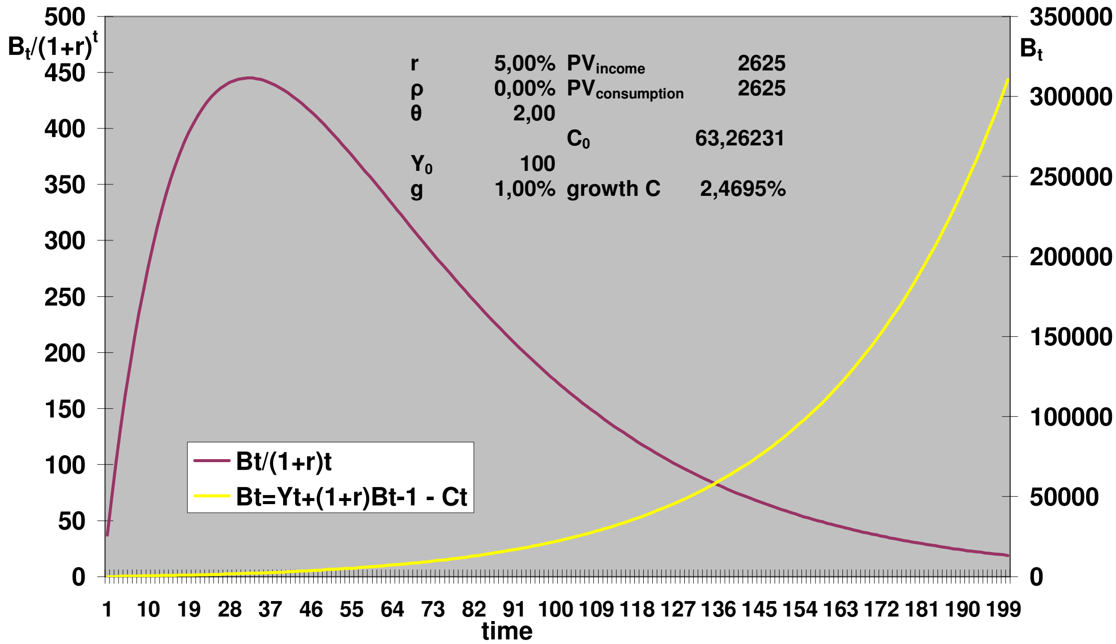
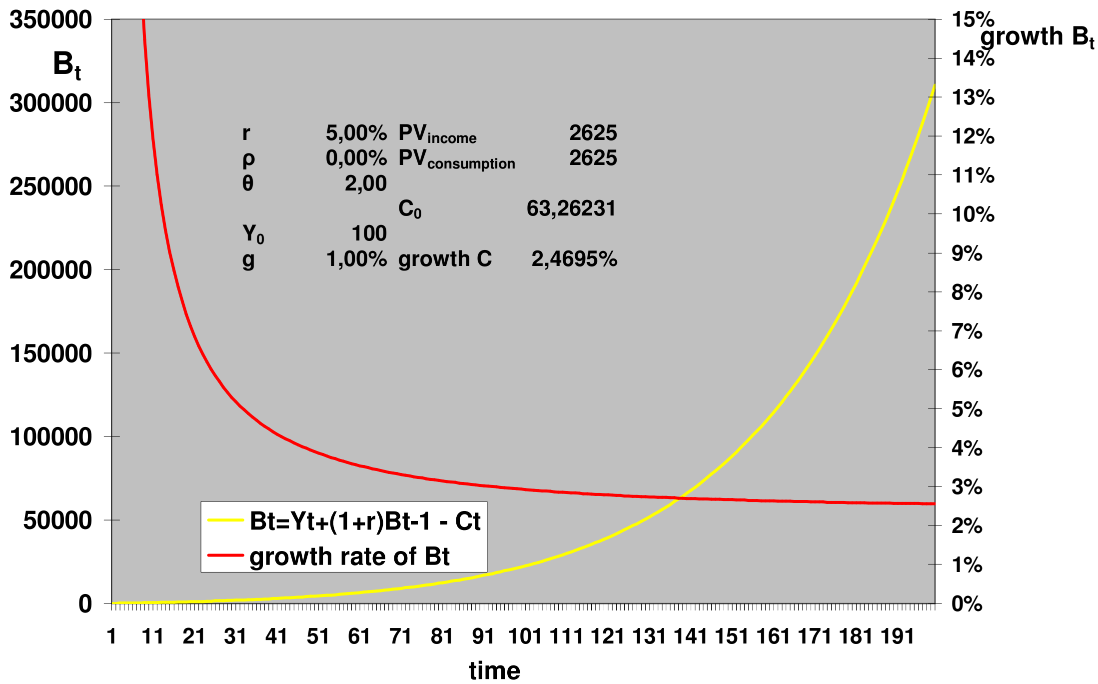


Figure no. 24_A4



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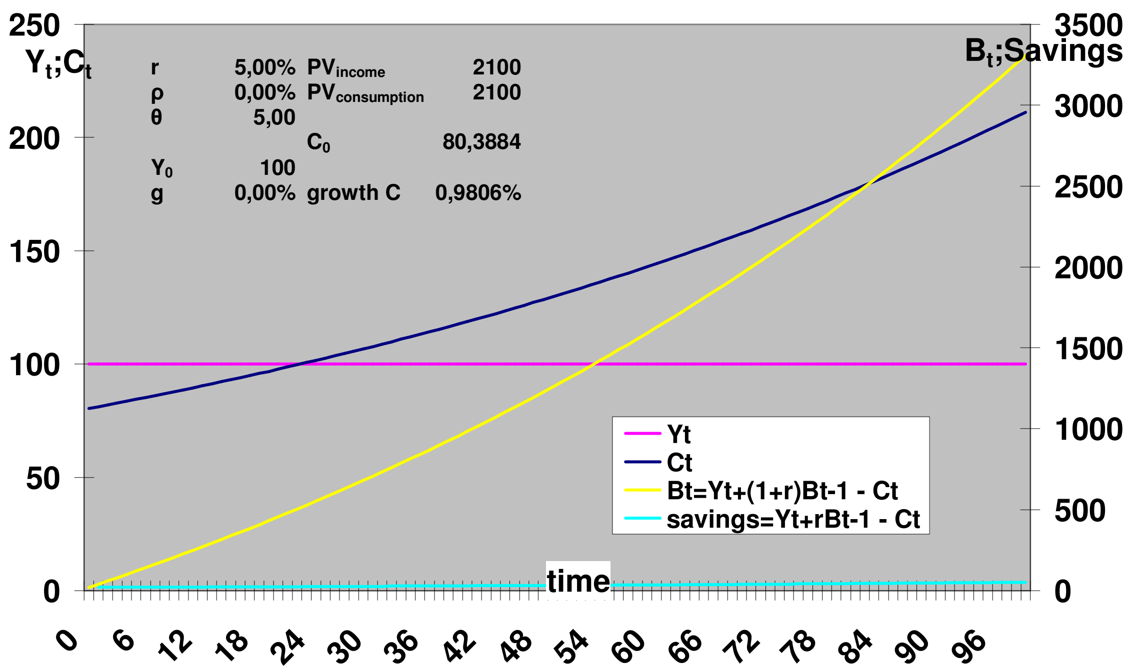

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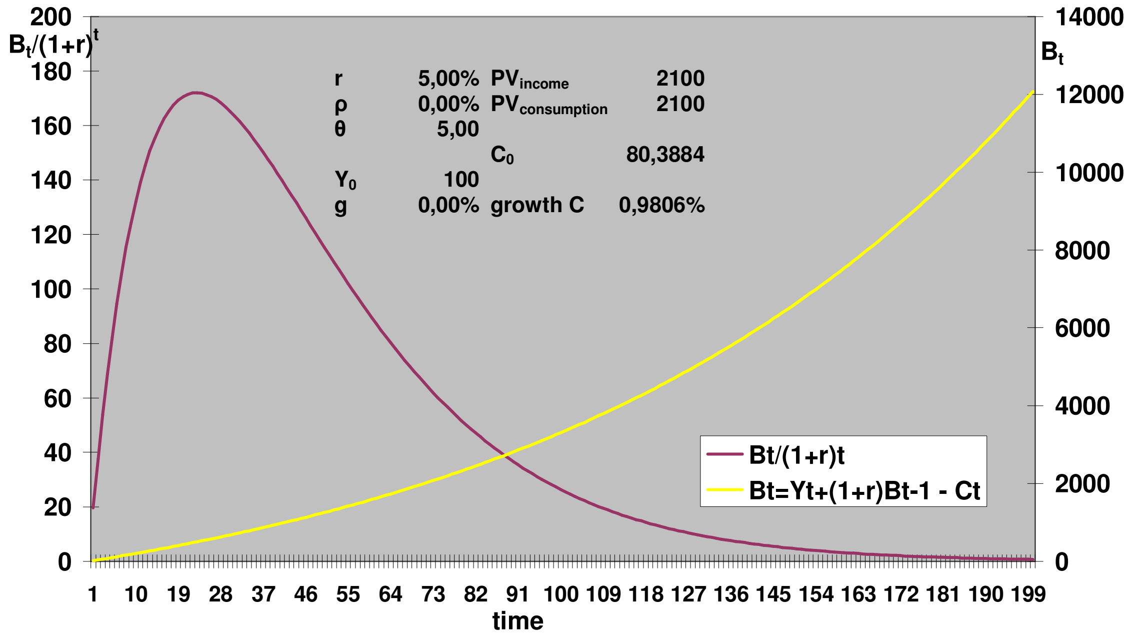
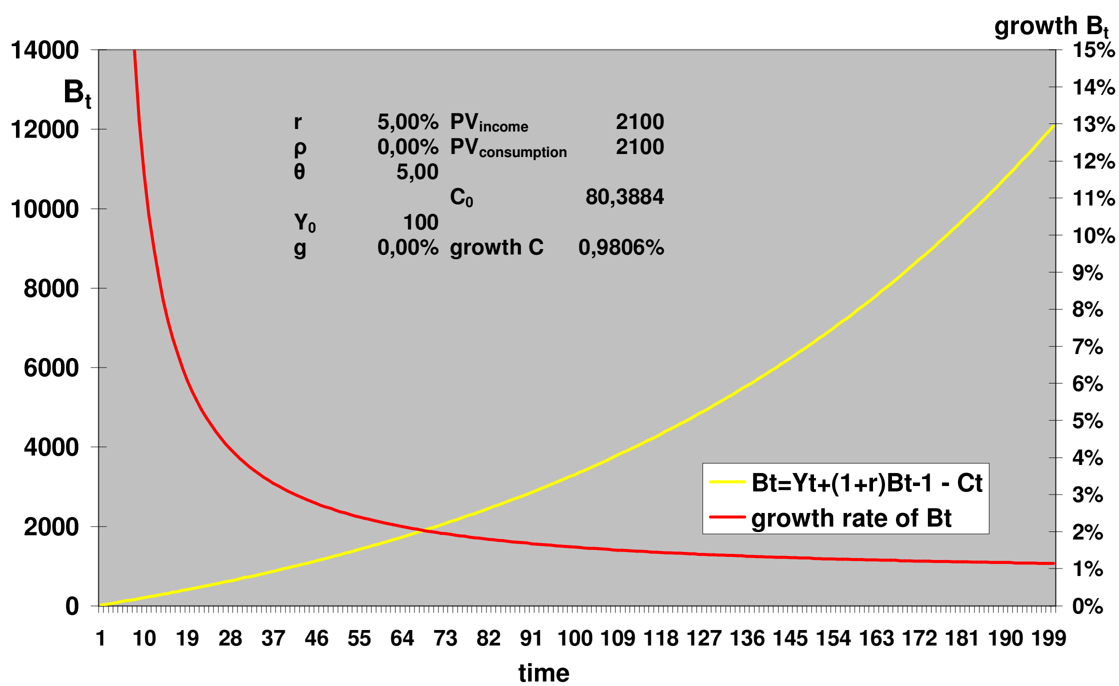


Figure no. 27_A4



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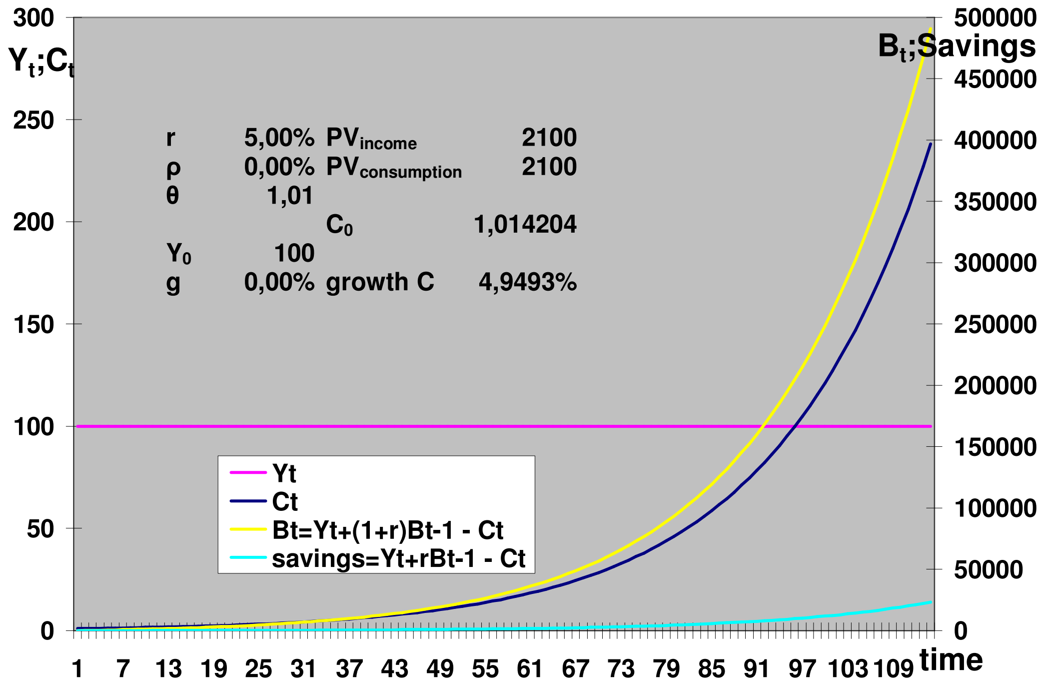

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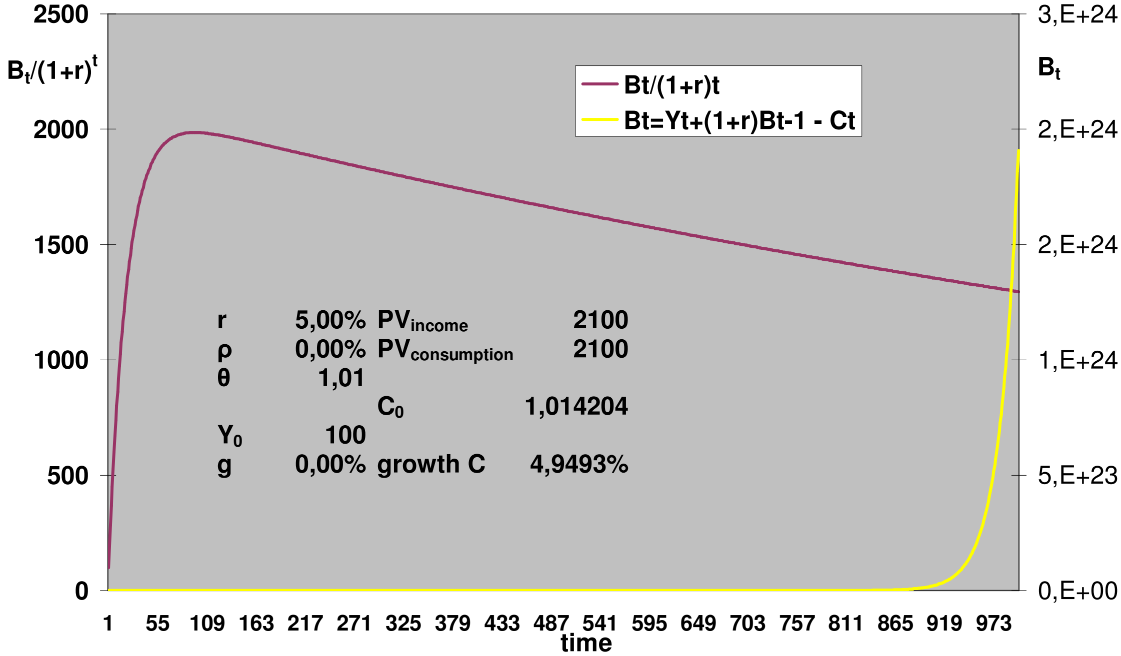

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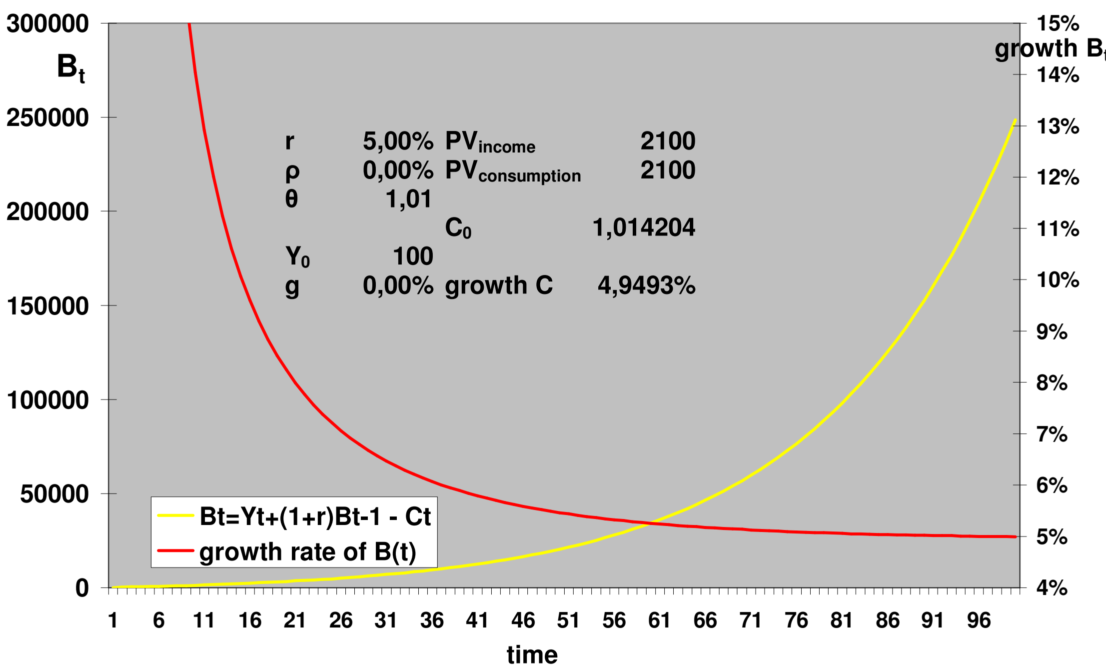
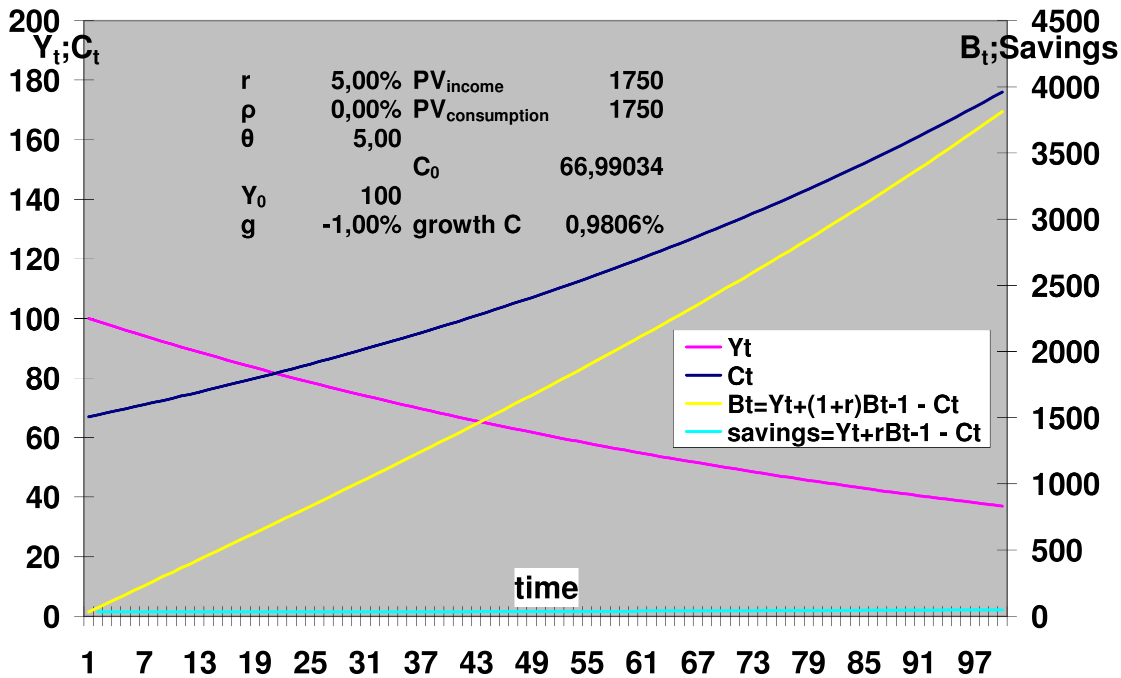


Figure no. 31_A4



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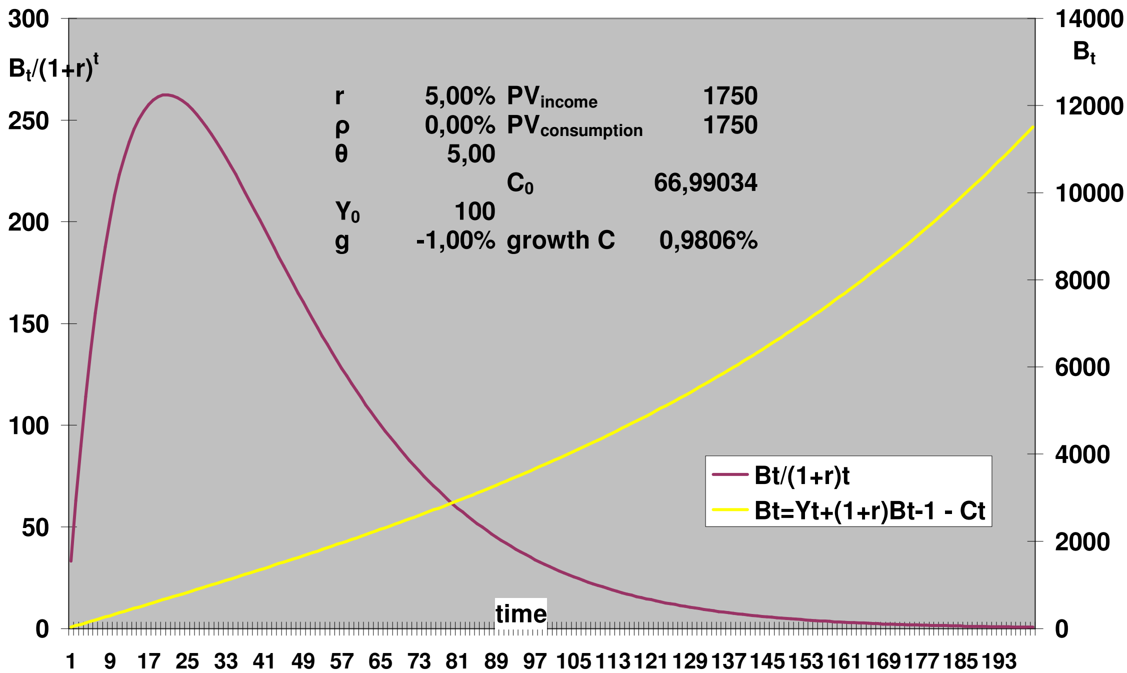

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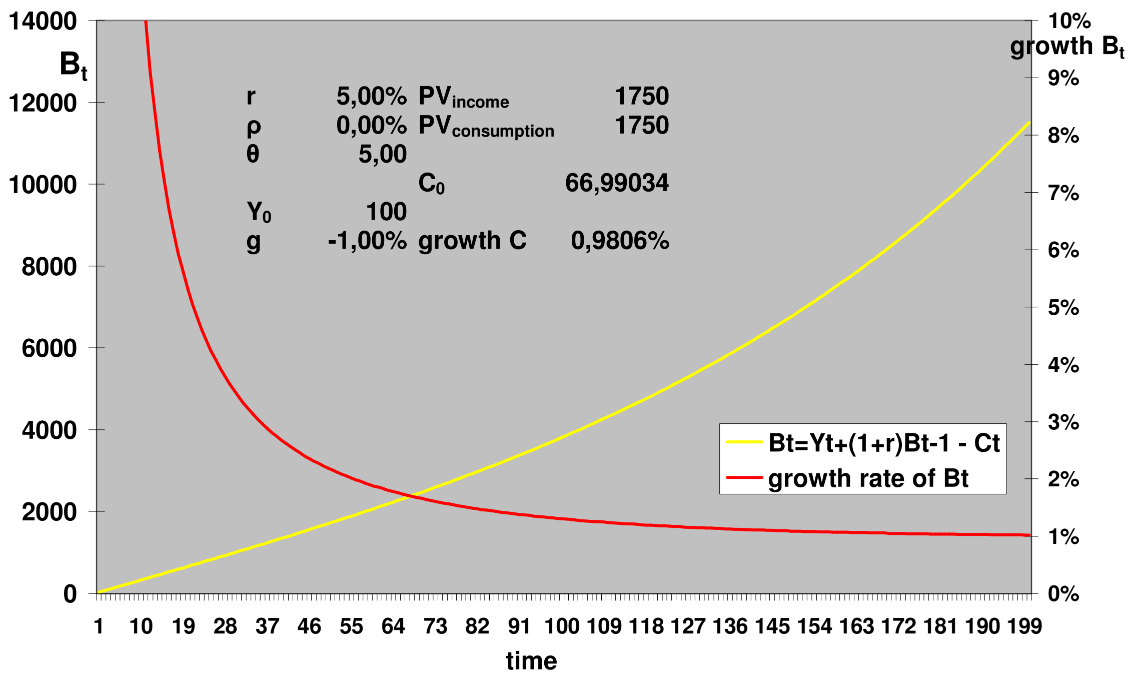

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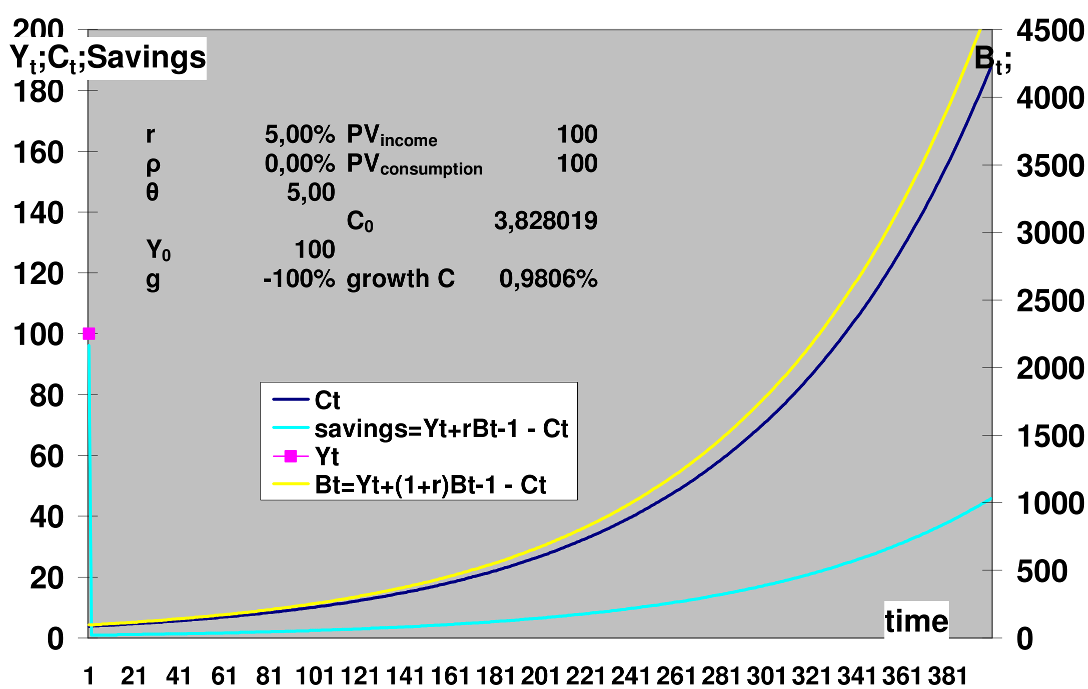

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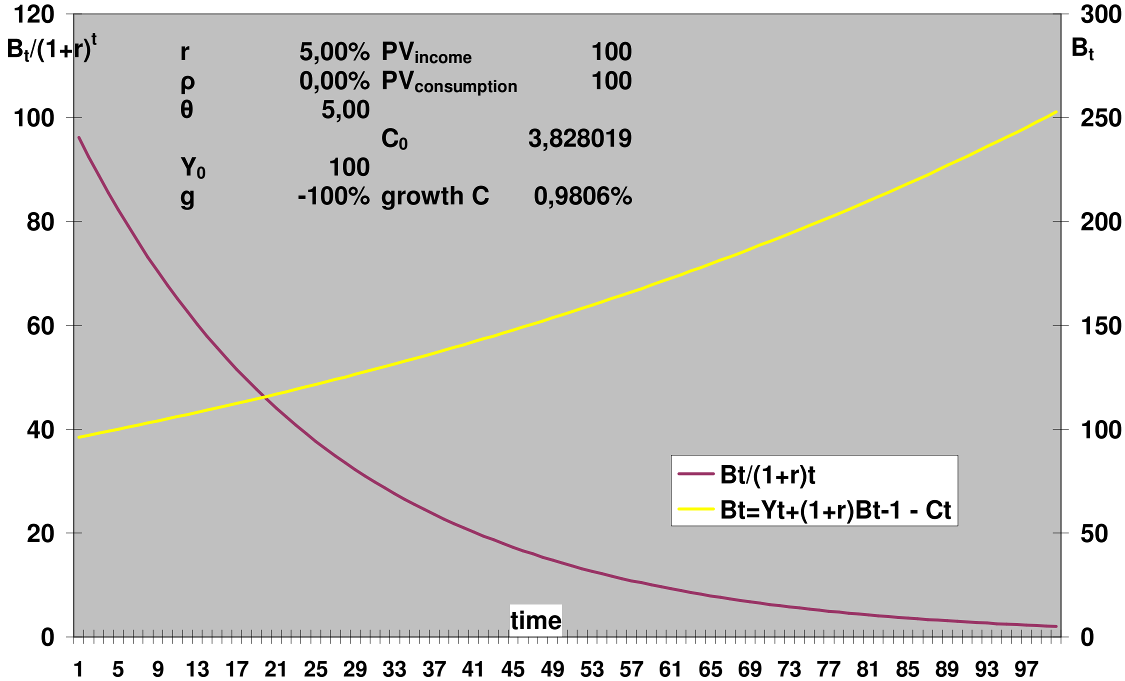

Figure no. 36_A4

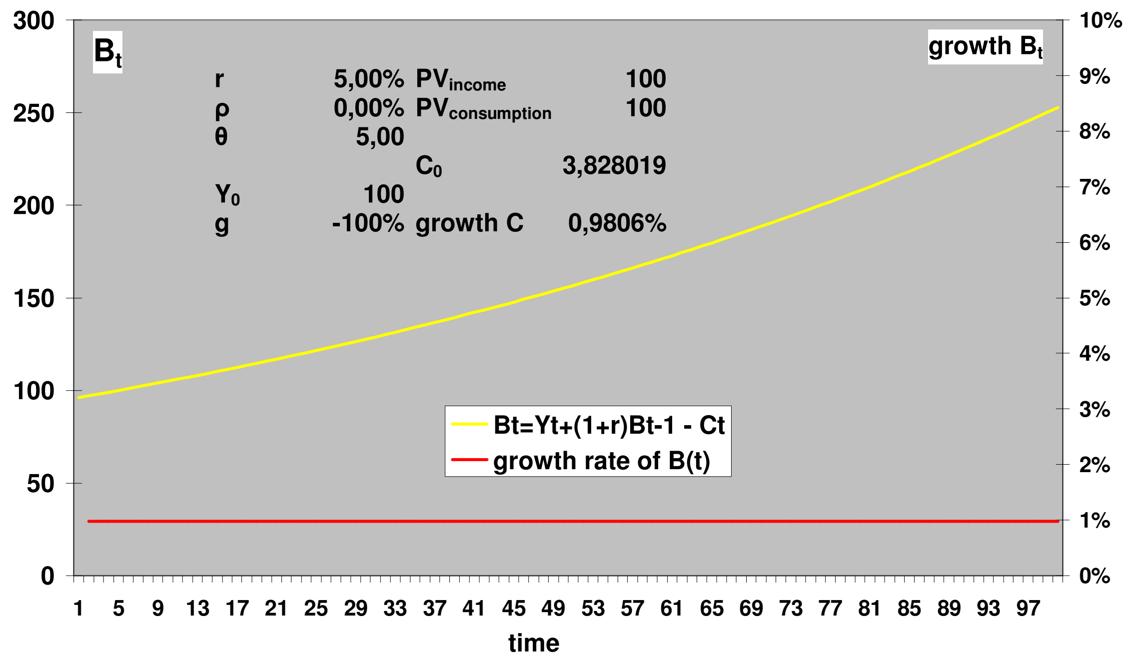

Figure no. 37_A4

Appendix 5

A) This appendix serves as a technical support for Section 5 in the main text and for Appendix 4. First, let us derive the intertemporal budget constrain in (A4_2) or (30) step by step. The present labour (i.e. non-capital) income earned in period 0 might be used for present consumption C_0 or it might be saved (B_0). If B_0 is positive, the individual is a saver in period 0. If it is negative the consumer is a debtor in period 0.

$$Y_0 = C_0 + B_0 \qquad \text{(A5\_1)}$$

In the next period, the accumulated saving increased by interest r_1 together with the labour income earned that period represent sources for consumption C_1 (see A5_2). If sources do not suffice, the individual must issue a debt at the size of $B_1 < 0$. If consumption falls short of the size of sources, he can buy a bond and B_1 is positive. It must be stressed that there is no necessary connection between B_0 and B_1. Thus, the individual might be a creditor in period 0 (i.e. $B_0 > 0$) and he may become a debtor in period 1 (i.e. $B_1 < 0$), if his consumption sufficiently exceeds his sources in that period (i.e. $Y_1 + B_0(1+r_1) < C_1$).

$$Y_1 + B_0(1+r_1) = C_1 + B_1 \qquad \text{(A5\_2)}$$

(A5_2) might be rewritten as:

$$Y_1 + r_1 B_0 - C_1 = B_1 - B_0 = \Delta B_1 \qquad \text{(A5\_3)}$$

Term ($Y_1 + r_1B_0$) represents his disposable income in period 1. The entire left-hand side stands for his saving as it shows a difference between disposable income and consumption. The right-hand side of the equation indicates a change in his net lending/borrowing position. Thus, saving in the given period has a crucial impact on the change in the lending (or borrowing) position. It should be stressed that we call saving only the difference between disposable income ($Y_1 + r_1B_0$) and C_1 (i.e. $B_1 - B_0 = \Delta B_1$) not the difference between his entire wealth ($Y_1 + (1+r_1)B_0$) and consumption C_1 (i.e. this difference would be B_1). The reason is that B_1 we will call a debt, if it is negative or "wealth before income" if it is positive.[46]

As can be seen, debt from the previous period is increased owing to the interest rate that is due in the subsequent period. Suppose that $B_0 < 0$. This debt will be $B_0(1+r_1)$ at the beginning of the next period. Provided that the entire labour income in period 1 (Y_1) is completely consumed (i.e. $Y_1 = C_1$), this debt can be easily financed by issuing a new debt ($B_1=(1+r_1)B_0$). As a result, debt will grow at the rate of the respective interest rate ($B_1/B_0 - 1 = r_1$), if the labour income in the given period is fully consumed. In such a case, we will say that debt is being rolled over to the future.

If the individual consumes only his disposable income ($C_1 = Y_1 + r_1B_0$), his debt in that period B_1 will be stabilized at the level of the previous period B_0 and the change in the borrowing position will be nil ($\Delta B_1=0$). Furthermore, the entire debt burden might be eliminated (i.e.$B_1=0$), only if consumption is low enough ($C_1 = Y_1 + (1+r_1)B_0 > 0$), but still positive.

[46] It might be said that saving is a flow concept and debt is a stock concept. Thus, saving must be a difference between two flow variables (disposable income and consumption), not between a stock variable (wealth) and a flow variable (consumption). As a result, equation (A5_2) indicates a link between stocks B_0 and B_1 that are adjusted by two flows - Y_1 and C_1. On the other hand, equation (A5_3) includes only flow variables – savings and a change in debt (i.e. a change in the stock variable).

Thus, a debt is not being rolled over by a simple issuing of a new debt, if consumption is lower than the labour income in the given period (i.e. $C_1 < Y_1$). From (A5_2) it can be seen that in such a case debt in the present value is partly reduced ($B_1/(1+r_1) > B_0$, i.e. $B_1/(1+r_1)$ is less negative than B_0), because $B_1 > B_0(1+r_1)$. However, the size of the debt (in the given period value) might increase (i.e. $B_1 < B_0$, thus $\Delta B_1 < 0$) if consumption is not lower than the entire disposable income ($C_1 > Y_1 + r_1 B_0$). Reduction of debt in the given period value therefore requires positive saving in that period ($C_1 < Y_1 + r_1 B_0$).

If we substitute B_0 from (A5_1) into (A5_2) and after a simple manipulation, we can write:

$$Y_0 + \frac{Y_1}{(1+r_1)} = C_0 + \frac{C_1}{(1+r_1)} + \frac{B_1}{(1+r_1)} \qquad \text{(A5\_4)}$$

In the two-period model, we implicitly assumed that all debts (in the present value) must be eventually settled, therefore $B_1/(1+r_1) \geq 0$. Such a condition might be called a No-Ponzi-Game condition for a two-period model. However, this also implies that $B_1 \geq 0$. At the same time, for a monotonically increasing utility function,[47] it would not be optimal to hold positive assets at the end of life (i.e. in period one), since consumption of all assets might increase utility. Thus, in the two-period model the last term in (A5_4) should disappear As a result, this equation simply states that the flow of consumption in the present value might not exceed (or better, is equal to) the flow of income in the present value.

However, let us now extend the time horizon to T. The budget constraint in period 2 (following the example of A5_2) can be written as:

$$Y_2 + B_1(1+r_2) = C_2 + B_2 \qquad \text{(A5\_5)}$$

Again, a debt in this period might be just rolled over to the future, if $Y_2 = C_2$. Thus, it will rise by the interest (rate), because $B_2=(1+r_2)B_1$. Of course, debt can increase even more, if that period consumption exceeds that period labour income. In such a case, the growth rate of debt will even exceed the rate that is implied by a simple roll-over strategy ($B_2/B_1 - 1 > r_2$). However, it might be also partly reduced in the present value if $C_2 < Y_2$. Furthermore, its size could be stabilized ($B_2 = B_1$; $\Delta B_2=0$), when consumption is reduced even more to the level of the disposable income ($C_2 = Y_2 + r_2 B_1$). In other words, consumption in that period must be low enough so that the consumer pays off (out of the given period labour income) the interest from the previous debt. And finally, debt could be completely eliminated, if consumption is depressed to such a level that $C_2 = Y_2 + (1+r_2)B_1$. However, it might happen that the resulting consumption is negative because $(1+r_2)B_1$ is too high (in negative value). In such a case, debt cannot be eliminated in that given period. Its elimination then requires a sufficient reduction of consumption also in subsequent periods, because consumption cannot fall below zero in any period.

Now, let us substitute (A5_4) into (A5_5) to obtain a similar idea as in (A5_4). A simple rearrangement of terms yields:

$$Y_0 + \frac{Y_1}{(1+r_1)} + \frac{Y_2}{(1+r_1)(1+r_2)} = C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_1)(1+r_2)} + \frac{B_2}{(1+r_1)(1+r_2)} \qquad \text{(A5\_6)}$$

If we generalize this procedure to T periods, (A5_6) becomes:

[47] I.e. for utility function having no satiation point. This might be concisely expressed as $u'(C)>0$ in the entire domain of positive real numbers (R_0^+).

$$C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_1)(1+r_2)} + ... + \frac{C_T}{(1+r_1)(1+r_2)...(1+r_T)} + \frac{B_T}{(1+r_1)(1+r_2)...(1+r_T)} =$$

$$= Y_0 + \frac{Y_1}{(1+r_1)} + \frac{Y_2}{(1+r_1)(1+r_2)} + ... + \frac{Y_T}{(1+r_1)(1+r_2)...(1+r_T)} \quad \text{(A5\_7)}$$

If the interest rate is constant over time, (A5_7) might be written as:

$$C_0 + \frac{1}{1+r}C_1 + \frac{1}{(1+r)^2}C_2 + \frac{1}{(1+r)^3}C_3 + ... + \frac{1}{(1+r)^T}C_T + \frac{1}{(1+r)^T}B_T =$$
$$= Y_0 + \frac{1}{1+r}Y_1 + \frac{1}{(1+r)^2}Y_2 + \frac{1}{(1+r)^3}Y_3 + ... + \frac{1}{(1+r)^T}Y_T \quad \text{(A5\_8)}$$

Olson and Bailey (1981:9) assumed the following form of the intertemporal budget constraint:

$$\sum_{t=0}^{T} \frac{Y_0^* - C_t}{(1+r)^t} = 0 \quad \text{(A5\_9)}$$

First, they assumed a constant level of income Y_0^* and constant interest rate r over time. Second, they implicitly imposed the requirement that all debts must be repaid in the end. Thus, $B_T/(1+r)^T \geq 0$. However, this also implies that $B_T \geq 0$, because $(1+r)^T$ is finite and positive. Again, this might be called a No-Ponzi-Game condition in a T-period model. At the same time, it cannot be optimal for an individual to hold any assets at time T. Thus, $B_T = 0$ and this term disappears in (B5_7) or (B5_8). As a result, (B5_7) or (B5_8) might be interpreted as follows: The discounted flow of consumption may not exceed (or better, is equal to) the discounted flow of income. Hence, we arrived at the now familiar form of the intertemporal budget constraint used in the main text (29 and 30) and in Appendix 4 (A4_2).

Furthermore, (B5_7) implies that debt must not be rolled-over till time T. Thus, C_t must not exceed Y_t in large enough number of periods, if some debt was issued in the past. But (B5_7) (or B5_8) requires even more. The net borrowing position must gradually decline ($\Delta B_t > 0$). Hence, consumption should be even lower than the disposable income and saving must be positive ($C_t < Y_t + r_t B_{t-1}$) in enough number of periods to reach zero debt at time T.

One technical note might be mentioned here, as it will be used in an infinite horizon model. (A5_5) at time T is:

$$Y_T + B_{T-1}(1+r_T) = C_T + B_T \quad \text{(A5\_10)}$$

This equation might be rearranged to:

$$\frac{Y_T - C_T}{B_{T-1}} + r_T = \frac{B_T - B_{T-1}}{B_{T-1}} \quad \text{(A5\_11)}$$

If the debt (or assets) at time T-1 (i.e. B_{T-1}) is big enough, the growth rate in debt (or assets) is almost unaffected by the difference between consumption and labour income and the major determinant is simply the interest rate. This holds for any time t, if B_{t-1} is large enough compared with Y_t and C_t.

Olson and Bailey (1981) extended their analysis to an infinite horizon. As T approaches infinity (A5_7) becomes:

$$C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_1)(1+r_2)} + ... = Y_0 + \frac{Y_1}{(1+r_1)} + \frac{Y_2}{(1+r_1)(1+r_2)} + ... \quad \text{(A5\_12)}$$

And if the interest rate is constant, then (see A5_8):

$$C_0 + \frac{1}{1+r}C_1 + \frac{1}{(1+r)^2}C_2 + \frac{1}{(1+r)^3}C_3 + ... = Y_0 + \frac{1}{1+r}Y_1 + \frac{1}{(1+r)^2}Y_2 + \frac{1}{(1+r)^3}Y_3 + ...$$

(A5_13)

The No-Ponzi-Game condition imposed on (A5_12) is (following our previous discussion):

$$\lim_{T\to\infty} \frac{B_T}{(1+r_1)(1+r_2)...} \geq 0 \qquad \text{(A5\_14)}$$

And the No-Ponzi-Game condition imposed on (A5_13) might be written as:

$$\lim_{T\to\infty} \frac{B_T}{(1+r)^T} \geq 0 \qquad \text{(A5\_15)}$$

The NPG in an infinite horizon, as well as in the finite horizon, states that the present value of assets cannot be negative.[48] However, in the infinite horizon this does not necessarily mean that B_T is non-negative too. The reason lies in the fact, that the denominator in (A5_14) or (A5_15) is infinite for any positive interest rate. Thus, NPG might be satisfied even for a negative B_T because discounting from infinity raises the denominator beyond all limits. Moreover, debt can even expand beyond all limits too (B_T can be negative infinity), if it reaches infinity "later" than the denominator in (A5_14) or (A5_15). Thus, even an infinite debt in infinity is consistent with the intertemporal budget constraints (A5_12) and (A5_13) and the one assumed by Olson and Bailey (1981).

To determine the growth rate at which the debt might be growing in the infinite horizon, apply (A5_11):

$$\lim_{T\to\infty} \left[\frac{Y_T - C_T}{B_{T-1}} + r_T \right] = \lim_{T\to\infty} \frac{\Delta B_T}{B_{T-1}} \qquad \text{(A5\_16)}$$

Thus, if B_{T-1} overcomes (Y_{T-1}- C_{T-1}) in the infinite horizon, the growth rate of debt might be very close to the interest rate and no saving is needed. Even so, the intertemporal budget constraint will be satisfied.

Let us now discuss the present value of the income flow, i.e. the right hand side of the intertemporal budget constraint (A5_13). We will assume for simplicity that the interest rate is constant over time. Furthermore, it will be required that the present value of income is finite even in the infinite horizon. The reason for this will become obvious below. Next, we assume that the (labour) income process is represented by the following equation:

[48] Economists usually say that NPG implies that debts cannot be rolled over forever (Romer:???)

$$Y_t = (1+g)Y_{t-1} \qquad \text{(A5\_17)}$$

Alternatively, (A5_17) implies:

$$Y_t = Y_0(1+g)^t \qquad \text{(A5\_18)}$$

Thus, income is growing at some given exogenous and constant rate. However, if g is zero, then the labour income is constant over time. If it is negative, labour income falls over time. Substituting (A5_18) into the right hand side of (A5_13), we get:

$$PV_{income} = Y_0 + \frac{Y_0\,(1+g)}{1+r} + \frac{Y_0\,(1+g)^2}{(1+r)^2} + \frac{Y_0\,(1+g)^3}{(1+r)^3} + \ldots \qquad \text{(A5\_19)}$$

A simple formula for the sum of the infinite geometric series gives us:

$$PV_{income} = Y_0\,\frac{1}{1-\frac{1+g}{1+r}} \qquad \text{(A5\_20)}$$

$$PV_{income} = Y_0\,\frac{1+r}{r-g} \qquad \text{(A5\_21)}$$

A finite PV in infinite horizon requires that the interest rate is greater than the growth rate in labour income. But this is exactly the condition for a dynamically efficient economy. Thus, r>g seems to be a reasonable assumption.

In Appendix 4, we also presented a formula for the PV of income, when the time horizon is finite. Consider the right-hand side of equation (A5_8) for zero debt at time T. Suppose that $Y_t=(1+g)^tY_0$. The PV of income is then:

$$PV_{income} = Y_0 + \frac{Y_0\,(1+g)}{1+r} + \frac{Y_0\,(1+g)^2}{(1+r)^2} + \frac{Y_0\,(1+g)^3}{(1+r)^3} + \ldots + \frac{Y_0\,(1+g)^T}{(1+r)^T} \qquad \text{(A5\_22)}$$

According to the formula of the sum of the finite geometric sequence, we may write:[49]

$$PV_{income} = Y_0\,\frac{1-\left(\frac{1+g}{1+r}\right)^{T+1}}{1-\frac{1+g}{1+r}} \qquad \text{(A5\_23)}$$

$$PV_{income} = \frac{(1+r)^{T+1}-(1+g)^{T+1}}{(1+r)^{T+1}}\,\frac{1+r}{1-g}\,Y_0 \qquad \text{(A5\_24)}$$

B) PV of utility!

In this part, we will examine the properties of the lifetime utility function in a more detail. Surprisingly, condition (A4_32) will be derived again. In this case, it will guarantee a convergence of the sum of instantaneous utilities.

(A4_1) for the CRRA might be written as:

[49] The exponent in the numerator must be (T+1) rather than T, because we start the sequence with T=0.

$$U=\sum_{t=0}^{\infty}\frac{\frac{C_t^{1-\theta}}{1-\theta}}{(1+\rho)^t}=\frac{C_0^{1-\theta}}{1-\theta}+\frac{1}{1+\rho}\frac{C_1^{1-\theta}}{1-\theta}+\frac{1}{(1+\rho)^2}\frac{C_2^{1-\theta}}{1-\theta}+\ldots \qquad \text{(A5\_25)}$$

As we have already demonstrated, the optimum path of consumption is described by (A4_9). Equation (A5_25) then yields:

$$U=\frac{C_0^{1-\theta}}{1-\theta}+\frac{1}{1+\rho}\frac{\left[C_0\left(\frac{1+r}{1+\rho}\right)^{1/\theta}\right]^{1-\theta}}{1-\theta}+\frac{1}{(1+\rho)^2}\frac{\left[C_0\left(\frac{1+r}{1+\rho}\right)^{2/\theta}\right]^{1-\theta}}{1-\theta}+\ldots \qquad \text{(A5\_26)}$$

$$U=\frac{C_0^{1-\theta}}{1-\theta}+\frac{C_0^{1-\theta}}{1-\theta}\frac{1}{1+\rho}\left(\frac{1+r}{1+\rho}\right)^{(1-\theta)/\theta}+\frac{C_0^{1-\theta}}{1-\theta}\frac{1}{(1+\rho)^2}\left[\left(\frac{1+r}{1+\rho}\right)^{(1-\theta)/\theta}\right]^2+\ldots \qquad \text{(A5\_27)}$$

The sum of this infinite series is:

$$U=\frac{C_0^{1-\theta}}{1-\theta}\frac{1}{1-\frac{1}{1+\rho}\left(\frac{1+r}{1+\rho}\right)^{(1-\theta)/\theta}} \qquad \text{(A5\_28)}$$

$$U=\frac{C_0^{1-\theta}}{1-\theta}\frac{1}{1-\frac{(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}} \qquad \text{(A5\_29)}$$

This sum converges if and only if condition (A4_32) holds. Thus, if the time preference is non-existent (ρ = 0), the marginal utility must be “dramatically diminishing” (θ>1).

Furthermore, by substituting (A4_37) into (A5_29), we get:

$$U=\frac{\left[Y_0\frac{1+r}{r-g}\frac{(1+\rho)^{1/\theta}-(1+r)^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta}}\right]^{1-\theta}}{1-\theta}\frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta}-(1+r)^{(1-\theta)/\theta}} \qquad \text{(A5\_30)}$$

$$U=\frac{\left(Y_0\frac{1+r}{r-g}\right)^{1-\theta}}{1-\theta}\frac{\left[(1+\rho)^{1/\theta}-(1+r)^{(1-\theta)/\theta}\right]^{1-\theta}}{(1+\rho)^{(1-\theta)/\theta}}\frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta}-(1+r)^{(1-\theta)/\theta}} \qquad \text{(A5\_31)}$$

$$U=\frac{\left(Y_0\frac{1+r}{r-g}\right)^{1-\theta}}{1-\theta}\frac{1+\rho}{\left[(1+\rho)^{1/\theta}-(1+r)^{(1-\theta)/\theta}\right]^{\theta}} \qquad \text{(A5\_32)}$$

As can be seen, lifetime utility increases with higher present labour income Y_0 and higher growth rate of the labour income g. It can be shown that the impact of r on U depends on the net lending/borrowing position of the consumer. If he is a lender (see A4_71) an increase in the interest rate will raise his lifetime utility. On the other hand, the borrowing position (A4_74) will result in the opposite outcome.

Equation (A5_29) (or A4_35) suggests that convergence of lifetime utility is consistent with only moderately diminishing marginal utility ($\theta < 1$), if people discount future utilities ($\rho > 0$). Figure no. 1_A5 outlines this idea. In panel a), a few discounted instantaneous utility functions are displayed. Supposing that the optimal consumption flow is constant (i.e. $r = \rho$), the resulting discounted instantaneous utility for any time period is plotted in panel b. As can be seen, the value of the utility decreases with increasing time horizon approaching zero in infinity. Thus, the lifetime utility represented as an area below this hypothetical curve converges to a finite number.

As has been already said, condition (A4_35) also requires θ greater than 1, if there is no discount on future utilities ($\rho = 0$). Notice that $\theta > 1$ implies that the instantaneous utility function lies below the horizontal axis (and it is more curved than in the previous figure).[50] Furthermore, positive interest rate and zero subjective discount rate thoroughly discussed in Appendix 4 lead to an increasing time shape of optimum consumption. As time elapses, consumption grows and instantaneous utility gradually approaches zero (see Figure no. 2_A5). Since there is no discounting, the instantaneous utility function in panel a) is at the same position. However, optimum consumption grows over time (see the horizontal axis in panel a). In panel b, the instantaneous utility is plotted for every given time period. As can be seen, the area above this hypothetical curve represents the lifetime utility and it converges to a finite number.

Hence, it can be said that the convergence of the lifetime utility might be induced either by sufficiently high time preference (in sense two), or if the time preference is low (or even zero) by a "dramatically diminishing marginal utility" and increasing path of consumption over time. This increasing stream is then generated by a positive difference between the interest rate and the subjective discount rate.

It should be stressed that from the point of view of our representative consumer, the real interest rate is an exogenous parameter. However, this does not hold for the economy as a whole. As was discussed in Appendix 4 and as is shown in section 5.1 of the main text, the interest rate r in a general equilibrium dynamic model should gradually reach $r^* = \rho + \theta g$ (see A4_43). At this specific point, consumption grows at the same rate as the labour income. There is also neither an eternal accumulation of assets nor debt (see A4_68 – A4_70). Thus, the positive difference between the real interest rate and the subjective discount rate leading to an increasing consumption over time is eventually caused by an increasing labour income ($g>0$). In standard growth models, increasing labour income is in turn a consequence of an exogenous technological progress. As a result, a positive gap between the real interest rate and the subjective discount rate is caused by advances in technologies. We can also see that if the time preference (ρ) is zero, only positive technological progress can induce positive real interest rate. Furthermore, if the subjective discount rate is zero, condition **(A4_85) or** (A4_32) is satisfied only for θ greater than 1.

[50] Nevertheless, a simple monotonic transformation may easily shift the utility function to positive quadrant leaving the behaviour of our consumer unaffected. Furthermore, marginal utility is positive and diminishing even for $\theta > 1$. The marginal rate of substitution between consumption in any time has typical properties as well. Thus, utility function posited in a negative quadrant poses no problem for our analysis.

We can conclude that the lifetime utility will converge even in the absence of explicit discounting of future utilities ($\rho = 0$), provided that the technological progress is positive ($g > 0$) and the utility function exhibits a sufficiently diminishing marginal utility ($\theta > 1$).

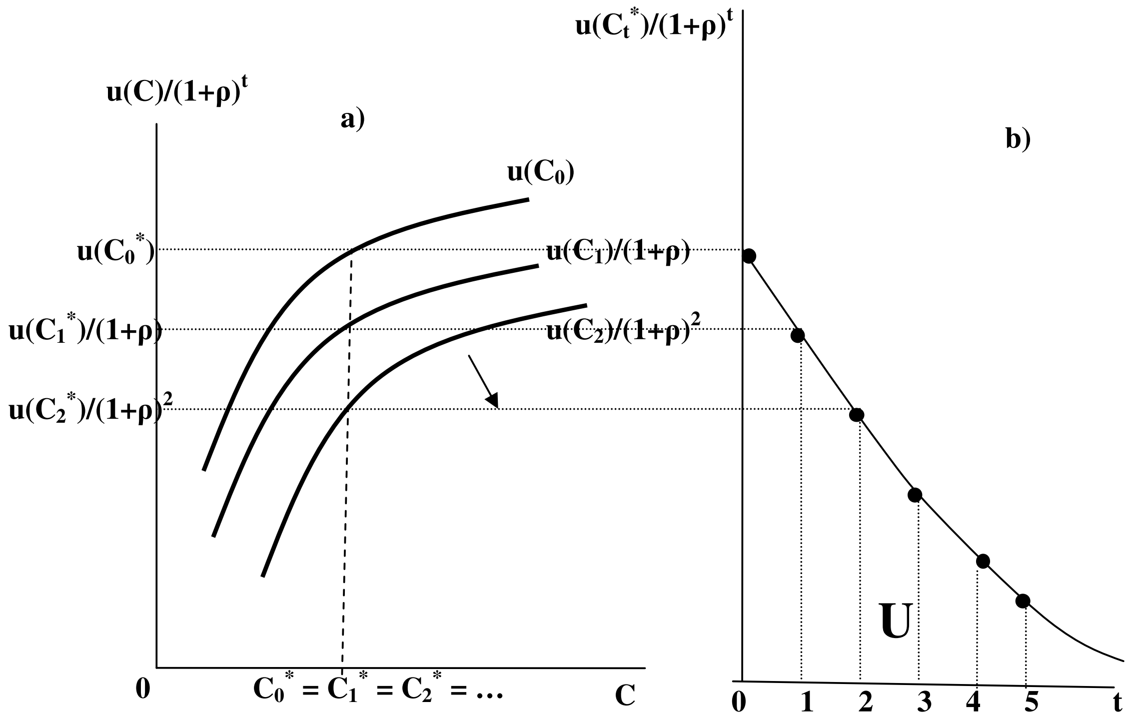

Figure no. 1_A5. CRRA utility function, $\rho = r > 0$ and $\theta < 1$.

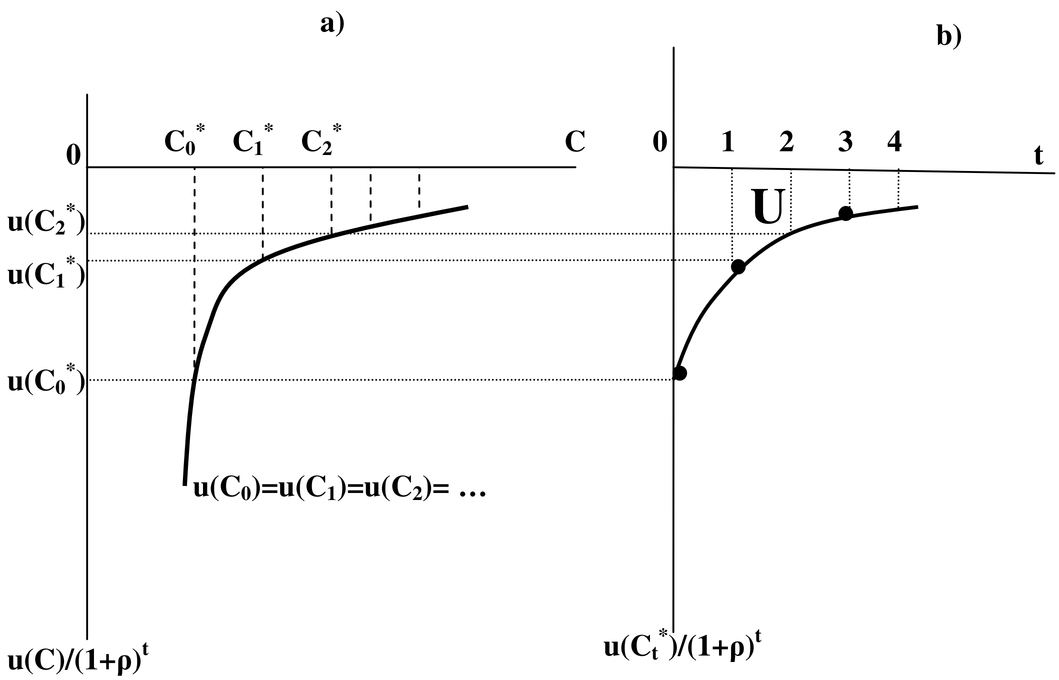

Figure no. 2_A5. CRRA utility function, $\rho=0$, $r > 0$ (e.g. $r = \theta g$), $\theta > 1$

Appendix 6

A) In this Appendix, we will solve the problem of a Fisherian (infinitely and finitely lived) shipwrecked sailor who is endowed with a fixed stock of hardtacks K.

Let us start with an infinite horizon. The objective of our sailor is to maximize his lifetime utility function expressed in continuous time as:[51]

$$U = \int_0^\infty e^{-\rho t} u(C(t))dt \qquad \text{(A6\_1)}$$

Subject to his resource (or budget) constraint:[52]

$$\int_0^\infty C(t)dt \leq K \qquad \text{(A6\_2)}$$

This condition states that lifetime consumption cannot exceed the initial endowment of hardtacks. Hardtacks have zero productivity, thus the MPK and the interest rate in this economy is zero. Furthermore, assuming the absence of the satiation point, (A6_2) should be satisfied with equality.

We will solve this dynamic problem with the help of calculus of variation.[53] Set up a Lagrangian function:

$$\int_0^\infty e^{-\rho t} u(C(t))dt + \lambda \left[K - \int_0^\infty C(t)dt \right] \qquad \text{(A6\_3)}$$

The solution of this dynamic problem should obey the Euler equation:

$$\frac{\partial F(\cdot)}{\partial C(t)} = \frac{d \dfrac{\partial F(\cdot)}{\partial \dot{C}(t)}}{dt} \qquad \text{(A6\_4)}$$

where $F(\cdot) = e^{-\rho t} u(C(t)) - \lambda C(t)$; and $\dot{C}(t) = \dfrac{dC(t)}{dt}$

Thus, using (A6_4) in solving (A6_3), we get:

$$e^{-\rho t} u'(C(t)) - \lambda = 0 \qquad \text{(A6\_5)}$$

(A6_5) simply states that in optimum, the discounted marginal utility of consumption in every period must be the same. To find the optimum growth rate in consumption, let us take logarithm of both sides of equation (A6_5):

$$-\rho t + \ln u'(C(t)) = \ln \lambda \qquad \text{(A6\_6)}$$

[51] This form was first presented in Samuelson (1937).
[52] See e.g. Strotz (1956:169).
[53] All the dynamic optimization methods presented here and in Appendix 7 can be found in Kamien, Schwarz (???)

And differentiate (A6_6) with respect to time:

$$-\rho + \frac{\frac{du'(C(t))}{dt}}{u'(C(t))} = 0 \qquad \text{(A6\_7)}$$

Equation (A6_7) reflects the requirement that in optimum, the growth rate in marginal utility of consumption must be equal to the subjective discount rate. To achieve this, consumption must fall over time accordingly.

Solving (A6_7), we get:

$$\frac{u''(C(t))\dot{C}(t)}{u'(C(t))} = \rho \qquad \text{(A6\_8)}$$

(A6_8) might be written as:

$$-\frac{\dot{C}(t)}{C(t)}\frac{u''(C(t))C(t)}{u'(C(t))} = -\rho \qquad \text{(A6\_9)}$$

$$\frac{\dot{C}(t)}{C(t)} = \frac{-\rho}{-\frac{u''(C(t))C(t)}{u'(C(t))}} \qquad \text{(A6\_10)}$$

Equation (A6_10) describes the optimum growth rate in consumption as a function of the subjective discount rate ρ and the Arrow-Pratt measure of the relative risk aversion v(C), which is represented by the denominator of (A6_10). For the CRRA utility function, v(C) is simply θ (see section B). As can be seen, the optimum consumption path must be definitely decreasing, because the interest rate is lower than the subjective discount rate (0 < ρ). The shape of this path then depends on the magnitude of the relative risk aversion (curvature of the utility function).

Furthermore (A6_7), is a simple differential equation that might be expressed as:

$$\frac{\mathrm{d}\ln\left[u'(C(t))\right]}{dt} = \rho \qquad \text{(A6\_11)}$$

Its solution is:

$$u'(C(t)) = Ae^{\rho t} \qquad \text{(A6\_12)}$$

A is an arbitrary constant that must be solved. For the CRRA utility function, (A6_12) might be written as:

$$C(t)^{-\theta} = Ae^{\rho t} \qquad \text{(A6\_13)}$$

$$C(t) = Ae^{-\rho t/\theta} \qquad \text{(A6\_14)}$$

At time 0, the optimum consumption is:

$$C(0) = A \qquad \text{(A6\_15)}$$

which must be, however, determined. To do that, let us insert (A6_15) into the resource constraint (A6_2):

$$\int_0^\infty C(0)e^{-\rho t/\theta}dt = K \qquad \text{(A6\_16)}$$

The solution of (A6_16) is:

$$\left[-C(0)\frac{1}{e^{\rho t/\theta}}\frac{\theta}{\rho}\right]_0^\infty = K \qquad \text{(A6\_17)}$$

$$C(0) = K\frac{\rho}{\theta} \qquad \text{(A6\_18)}$$

Because present consumption cannot exceed the initial endowment, θ must be greater than ρ. Substituting (A6_18) into (A6_14) we obtain the optimal path of consumption in the infinite horizon:

$$C(t) = K\frac{\rho}{\theta}e^{-\rho t/\theta} \qquad \text{(A6\_19)}$$

To obtain the optimum in the finite horizon, (A6_17) might be written as:

$$\left[-C(0)\frac{1}{e^{\rho t/\theta}}\frac{\theta}{\rho}\right]_0^T = K \qquad \text{(A6\_20)}$$

Optimal C(0) is then:

$$C(0)\frac{\theta}{\rho}\left(1-\frac{1}{e^{\rho \mathrm{T}/\theta}}\right) = K \qquad \text{(A6\_21)}$$

$$C(0) = \frac{\rho}{\theta}\frac{e^{(\rho/\theta)\mathrm{T}}}{e^{(\rho/\theta)\mathrm{T}}-1}K \qquad \text{(A6\_22)}$$

Present consumption must not exceed the initial endowment. (A6_22) thus implies:

$$\frac{e^{(\rho/\theta)\mathrm{T}}}{e^{(\rho/\theta)\mathrm{T}}-1} \leq \frac{\theta}{\rho} \qquad \text{(A6\_23)}$$

$$\frac{\rho}{\theta} \leq \frac{e^{(\rho/\theta)\mathrm{T}}-1}{e^{(\rho/\theta)\mathrm{T}}} \qquad \text{(A6\_24)}$$

$$\frac{\rho}{\theta} \leq 1-\frac{1}{e^{(\rho/\theta)\mathrm{T}}} \qquad \text{(A6\_25)}$$

$$\frac{1}{e^{(\rho/\theta)\mathrm{T}}} \leq 1-\frac{\rho}{\theta} \qquad \text{(A6\_26)}$$

$$-\frac{\rho}{\theta}T \leq \ln(1-\frac{\rho}{\theta}) \qquad \text{(A6\_27)}$$

If ρ/θ is a small number then:

$$T \geq 1 \qquad \text{(A6\_28)}$$

If it is not, (A6_27) yields:

$$T \geq -\frac{\theta}{\rho}\ln(1-\frac{\rho}{\theta}) \qquad \text{(A6\_29)}$$

$$T \geq \frac{\theta}{\rho}\ln\frac{\theta}{\theta-\rho} \qquad \text{(A6\_30)}$$

Again, θ must be greater than ρ. However, there is also a very weak restriction on the length of the planning horizon T.

To obtain the optimal path of consumption in the finite horizon T, let us substitute (A6_22) into (A6_14):

$$C(t) = \frac{\rho}{\theta}\frac{e^{(\rho/\theta)\mathrm{T}}}{e^{(\rho/\theta)\mathrm{T}}-1}Ke^{-\rho t/\theta} \qquad \text{(A6\_31)}$$

$$C(t) = \frac{\rho}{\theta}\frac{e^{\rho(\mathrm{T}-t)/\theta}}{e^{(\rho/\theta)\mathrm{T}}-1}K \qquad \text{(A6\_32)}$$

Appendix 7

In this appendix, we will derive a simple textbook Ramsey-Cass-Koopmans model that will complete our comparison with the Misesian theory of interest on one hand and the Hayekian approach on the other.

Aggregate output in a given period depends on capital, labour and labour-augmenting technological progress in the same period:

$$Y(t) = F(K(t), A(t)L(t)) \qquad \text{(A7\_1)}$$

Capital is essential in production, thus:[54]

$$F(0, A(t)L(t)) = 0 \qquad \text{(A7\_2)}$$

Marginal product of capital is positive and diminishing for all levels of capital:

$$F_K > 0; F_{KK} < 0 \qquad \text{(A7\_3)}$$

And it satisfies usual Inada conditions that guarantee an interior steady state:

$$\lim_{K\to 0} F_K = \infty \qquad \text{(A7\_4)}$$

$$\lim_{K\to \infty} F_K = 0 \qquad \text{(A7\_5)}$$

The labour force grows exogenously at the rate of n (which is the "force" of the labour force), L(0) is the initial size of the labour force:

$$\frac{\dot{L}(t)}{L(t)} = n \text{; thus } L(t) = L(0)e^{nt} \qquad \text{(A7\_6)}$$

Similar idea holds also for technological progress growing at rate g:

$$\frac{\dot{A}(t)}{A(t)} = g \text{; hence } A(t) = A(0)e^{gt} \qquad \text{(A7\_7)}$$

Capital accumulation is described by a simple neoclassical (and rather "non-Austrian") law of motion of capital:

$$\dot{K}(t) = sY(t) - \delta K(t) \qquad \text{(A7\_8)}$$

(A7_8) states that instantaneous change in capital stock dK(t)/dt (i.e. net investment) depends on the difference between saving sY(t), which is always equal to gross investment in this growth theory, and depreciation δK(t), where δ is the exogenous depreciation rate.[55]

Production function exhibits constant returns to scale, so (A7_1) might be divided by the amount of effective labour A(t)L(t) to stationarize the model. Thus:

$$\frac{Y(t)}{A(t)L(t)} = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \qquad \text{(A7\_9)}$$

[54] Assumptions (A7_2) - (A7_5) hold also for labour. See Acemoglu (???:???)
[55] See Hayek (Maintenance of Capital) for a discussion about usual approach to depreciation and the poor legitimacy of the theoretical separation of pure replacement of capital and net investment.

By defining y=Y/AL, k=K/AL and f(k)=F(k,1), we get an intensive form of the production function (A7_10). Due to the CRS assumption, the size of the economy does not matter.

$$y(t) = f(k(t)) \qquad \text{(A7\_10)}$$

Furthermore, all assumptions about the extensive form are inherited also by the intensive form. Thus:

$$f(0) = 0 \qquad \text{(A7\_10b)}$$

$$f'(k) > 0 \qquad \text{(A7\_10c)}$$

$$f''(k) < 0 \qquad \text{(A7\_10d)}$$

$$\lim_{k \to 0} f'(k) = \infty \qquad \text{(A7\_10e)}$$

$$\lim_{k \to \infty} f'(k) = 0 \qquad \text{(A7\_10f)}$$

We will obviously assume that capital cannot be negative, hence:

$$k(t) \geq 0 \qquad \text{(A7\_10g)}$$

The law of motion of capital in the intensive form is derived from:

$$\dot{k}(t) = \frac{d\left(\dfrac{K(t)}{A(t)L(t)}\right)}{dt} = \frac{\dot{K}(t)A(t)L(t) - K(t)\left[\dot{A}(t)L(t) - A(t)\dot{L}(t)\right]}{\left[A(t)L(t)\right]^2} = \frac{sY(t) - \delta.K(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{A}(t)}{A(t)} -$$

$$- \frac{K(t)}{A(t)L(t)}\frac{\dot{L}(t)}{L(t)} = sy(t) - (\delta + g + n)k(t) = sf(k(t)) - (\delta + g + n)k(t) = f(k(t)) - c(t) - (\delta + g + n)k(t)$$

(A7_11)

c(t) denotes consumption per effective worker c=C/A. The last part of (A7_11) uses the fact that saving sf(k(t)) is the difference between output y(t) and consumption c(t).

The objective of a representative infinitely lived dynasty (household) is to maximize lifetime utility given by:

$$U = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt \qquad \text{(A7\_12)}$$

L(t)/H is the number of members of this dynasty at time t, since H measures the number of dynasties (households) in the economy, which is fixed by assumption, and the L(t) stands for the size of the labour force at time t. Every member of the household works and offers one unit of labour in-elastically in every period.

(A7_12) might be easily transformed to an intensive form:

$$U = \int_0^{\infty} e^{-\rho t} \frac{[A(t)c(t)]^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt = \int_0^{\infty} e^{-\rho t} \frac{[A(0)e^{gt}c(t)]^{1-\theta}}{1-\theta} \frac{L(0)e^{nt}}{H} dt =$$
$$= \frac{A(0)^{1-\theta} L(0)}{H} \int_0^{\infty} e^{-[\rho - n - (1-\theta)g]t} \frac{c(t)^{1-\theta}}{1-\theta} dt \qquad \text{(A7\_13)}$$

The convergence of this integral requires that:[56]

$$\rho - n - (1-\theta)g > 0 \qquad \text{(A7\_14)}$$

Suppose, along with Hayek (1941),[57] that a central planner is trying to maximize lifetime utility of a representative dynasty (A7_13) subject to the resource constraint of the economy (A7_11) and the non-negativity constraint (A7_10g). Let us set up a simple (present value) Hamiltonian:[58]

$$H = \frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho - n - (1-\theta)g]t} \frac{c(t)^{1-\theta}}{1-\theta} + \lambda(t)[f(k(t)) - c(t) - (\delta + g + n)k(t)] \qquad \text{(A7\_15)}$$

The first order conditions are:

$$\frac{\partial H}{\partial c(t)} = \frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho - n - (1-\theta)g]t} c(t)^{-\theta} - \lambda(t) = 0 \qquad \text{(A7\_16)}$$

$$\frac{\partial H}{\partial k(t)} = \lambda(t)[f'(k(t)) - (n + g + \delta)] = -\dot{\lambda}(t) \qquad \text{(A7\_17)}$$

And the transversality condition is:

$$\lim_{t \to \infty} \lambda(t)k(t) = 0 \qquad \text{(A7\_17b)}$$

This condition requires that the shadow price of capital λ(t) in the infinite horizon is zero for the social planner or that the capital per effective worker k(t) in infinity is zero. We will see that k*>0 is the steady of this model, thus k(t) is non-zero in infinity. Hence, we require λ(t)=0 in infinity.

Condition (A7_17) implies that:

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[56] See Appendix 4 for a thorough discussion about a similar condition. At the steady state, c(t) is constant, hence the first term inside the integral must approach zero in infinite time.

[57] Surprisingly, Hayek developed part of his theory of capital on the assumption of a benevolent central planner. We will follow this assumption in this section. Nonetheless, the decentralized economy solution of the model is exactly the same as for the central planner. See e.g. Romer (???) or Barro, Sala-i-Martin (???)

[58] This procedure might be found e.g. in Blanchard, Fischer (1991??:39-41)

$$\frac{-\dot{\lambda}(t)}{\lambda(t)} = [f'(k(t)) - (n+g+\delta)] \qquad \text{(A7\_18)}$$

At the same time, (A7_16) might be transformed to a similar differential equation by taking logarithm of both sides:

$$\ln \frac{A(0)^{1-\theta} L(0)}{H} - [\rho - n - (1-\theta) g]t - \theta \ln c(t) = \ln \lambda(t) \qquad \text{(A7\_19)}$$

And differentiating (A7_19) with respect to time:

$$-[\rho - n - (1-\theta) g] - \theta \frac{\dot{c}(t)}{c(t)} = \frac{\dot{\lambda}(t)}{\lambda(t)} \qquad \text{(A7\_20)}$$

(A7_20) and (A7_18) then imply an optimum growth rate of consumption:

$$[\rho - n - (1-\theta) g] + \theta \frac{\dot{c}(t)}{c(t)} = f'(k(t)) - (n+g+\delta) \qquad \text{(A7\_21)}$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} \qquad \text{(A7\_22)}$$

Realizing that c=C/A, equation (A7_22) implies that the optimum growth rate of consumption of one single member of the dynasty is as follows:

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{c}(t)}{c(t)} + \frac{\dot{A}(t)}{A(t)} = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} + g = \frac{f'(k(t)) - \delta - \rho}{\theta} \qquad \text{(A7\_22\_B)}$$

Equation (A7_22_B) (or A7_22) is the Euler equation for this model, known also as the Keynes-Ramsey rule. It states that the optimum growth rate of consumption depends positively on the net return to capital (f´(k(t)) – δ) and negatively on the subjective discount rate ρ. The coefficient of the relative risk aversion θ modifies the optimum response of the growth rate of consumption to the difference between net return to capital and subjective discount rate. We will see that at the steady state, c(t) is constant, thus consumption C(t) grows at the rate of technological progress g.

Now, we can determine the set of parameters that will satisfy the transversality condition (A7_17b). Equation (A7_18) implies that:

$$\lambda(t) = \lambda(0) e^{-[f'(k(t)) - (n+g+\delta)].t} \qquad \text{(A7\_22b)}$$

(A7_17b) then results in:

$$\lim_{t \to \infty} \lambda(t) k(t) = \lim_{t \to \infty} \lambda(0) e^{-[f'(k(t)) - (n+g+\delta)].t} k(t) = 0 \qquad \text{(A7\_22c)}$$

As we will see, the steady state of capital per effective worker k(t) in this model is positive ($k^* > 0$). Condition (A7_22c) thus requires:

$f'(k^*)-\delta > n+g$ (A7_23d)

(((Later, it will be derived (see A7_46) that the real interest rate equals the net return to capital and that at the steady state of $k(t)=k^*$, n+g is equal to the growth rate in real GDP (see A7_???). Hence, (A7_23d) requires that the real interest rate must be greater than the growth rate in real GDP. This in turn implies that the economy must be dynamically efficient at the steady state (on the balanced growth path) of this model.)))

Transversality condition (A7_17b) can be also expressed (using A7_16) as:

$$\lim_{t\to\infty} \frac{A(0)^{1-\theta} L(0)}{H} e^{-[\rho-n-(1-\theta)g]t} c(t)^{-\theta} k(t) = 0 \qquad \text{(A7\_23e)}$$

Since at the steady state of this model, c^* and k^* are positive (i.e. $c(t)=c^*>0$ and $k(t)=k^*>0$ as time goes to infinity) condition (A7_23e) requires condition (A7_14), i.e. $\rho-n-(1-\theta)g>0$, again to be valid.

A more straightforward economic interpretation of (A7_23e) might be obtained, if it is rearranged as:

$$\lim_{t\to\infty} \frac{A(0)e^{g t} A(0)^{-\theta} e^{-\theta g t} L(0) e^{n t}}{H} e^{-\rho t} \frac{C(t)^{-\theta}}{A(t)^{-\theta}} \frac{K(t)}{A(t)L(t)} = 0 \qquad \text{(A7\_23f)}$$

$$\lim_{t\to\infty} e^{-\rho t} C(t)^{-\theta} \frac{K(t)}{H} = 0 \qquad \text{(A7\_23g)}$$

(A7_23g) states that in the infinite horizon, the discounted marginal utility of consumption $e^{-\rho t}C(t)^{-\theta}$ multiplied by the amount of capital per household must be zero. This might be achieved by several ways. Either capital is consumed at the terminal date,[59] or consumption is infinite in the infinite future depressing marginal utility $C(t)^{-\theta}$ to zero, or the subjective discount rate is positive ($\rho>0$).

However, since capital K(t) is not zero (quite the contrary - it is infinite, growing at the rate of n + g) at the steady state of $k(t)=k^*$, the zero value of the shadow price of capital $\lambda(t)$ in infinity is achieved either by the fact that personal discounting sufficiently depresses the importance of capital to zero or that unbounded consumption in the infinite horizon places zero value to the additional unit of capital. Moreover, these two tendencies must more than offset the eternal growth in capital. Thus, condition (A7_14) reflects this requirement, since it can be rewritten as:

$$\rho + \theta g - n - g > 0 \qquad \text{(A7\_23h)}$$

At the steady state of this model (balanced growth path), the growth rate in capital n+g must fall short of the sum of the subjective discount rate ρ and the growth rate in consumption g modified by parameter θ, which reflects the rapidity at which the marginal utility diminishes. As can be seen, the higher this rapidity the higher the chance that condition (A7_23h) will be satisfied provided that the technological progress is positive.

So far, we considered the problem of a benevolent central planner. However, our main task is to explore the behaviour of the interest rate, which is a market economy phenomenon.

[59] This is usually implied in the finite horizon version of this model. See e.g. Shell (1969).

Nonetheless, the solution for a decentralized economy will be exactly the same. This is a direct proof that the decentralized market economy in the RCK model finds its dynamic equilibrium at the same point as would be chosen by a benevolent social planner.

A thorough discussion in appendix 5 gave us a clear idea of the accumulation of assets of a consumer in the discrete time (see A5_3):

$$Y_{t+1} + r_{t+1}B_t - C_{t+1} = B_{t+1} - B_t = \Delta B_{t+1} \qquad \text{(A7\_23)}$$

In continuous time (where the difference between period t and t+1 is infinitely small), equation (A7_23) might be written as:[60]

$$\dot{B}(t) = W(t) - C(t) + r(t)B(t) \qquad \text{(A7\_24)}$$

However, in this section, we consider a problem of the whole dynasty that is growing over time at the rate of n. As a result, equation (A7_24) will be slightly modified, since we will consider assets (or debt) of the entire household, not just one member. A similar idea as developed in the previous two equations will give us:

$$\dot{B}_H(t) = W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} + r(t)B_H(t) \qquad \text{(A7\_25)}$$

$B_H(t)$ represents the assets accumulated by a representative household till time t. L(t)/H is the number of members of each household, therefore W(t)L(t)/H is the entire labour income of the household at time t and C(t)L(t)/H is the total consumption of household in the same period.

Each household is, however, restricted by the credit market. It simply cannot roll over its debt forever (see a thorough discussion in Appendix 5). As a result, we have to impose the No-Ponzi-Game condition even in the continuous time model (see equation A5_14). Again, it requires that the debt of each household cannot asymptotically grow faster than the interest rate (Blanchard,Fischer 1989:49; Romer 2006:52):

$$\lim_{t\to\infty} e^{-R(t)}B_H(t) \geq 0 \qquad \text{(A7\_25b)}$$

$$R(t) = \int_0^t r(\tau)d\tau \qquad \text{(A7\_25c)}$$

$e^{-R(t)}$ is a continuous time version of discounting presented in A5_14. It might be derived from the idea that:

$$\frac{1}{(1+\mathrm{r}_1)(1+\mathrm{r}_2)...(1+\mathrm{r}_t)} \approx \exp[-(r_1 + r_2 + ...r_t)] = \exp(-\sum_{\tau=0}^{t} r_\tau) = \exp(-R_t) \qquad \text{(A7\_25d)}$$

Thus, $e^{-R(t)}$ is a continuous time version of (A7_25d). As can be clearly seen, R(t) in equation (A7_25c) adds all instantaneous rates of interest (i.e. “forces” of interest) from time 0 to time

[60] For the labour income of one individual in the general equilibrium model, we will use expression W(t)l(t) rather than Y(t). However, each member of household offers just 1 unit of labour in every period, i.e. l(t)=1. Thus, we get W(t) as the labour income of each individual at time t. However, all variables in (A7_24) are of instantaneous nature. See section C of this appendix to derive (A7_24) from (A7_23).

t. Yet, summing in the continuous time is performed by the integral rather than by the sum. Hence, $e^{-R(t)}$ represents the idea of (continuous) discounting, whose discrete time version can be seen in the first part of expression (A7_25d).

Furthermore, it can be easily seen that the amount of assets of the whole household is simply the amount of assets of one member times the number of members of each household:

$$B_H(t) = \frac{L(t)}{H} B(t) \qquad \text{(A7\_26)}$$

Thus, assets of one member are represented by:

$$B(t) = H \frac{B_H(t)}{L(t)} \qquad \text{(A7\_27)}$$

As a result, the law of motion of assets of one member is given by:

$$\dot{B}(t) = H \frac{\dot{B}_H(t)L(t) - B_H(t)\dot{L}(t)}{L^2(t)} = H\left[\frac{\dot{B}_H(t)}{L(t)} - \frac{B_H(t)}{L(t)}\frac{\dot{L}(t)}{L(t)}\right] = H\left[\frac{\dot{B}_H(t)}{L(t)} - n\frac{B_H(t)}{L(t)}\right] \text{(A7\_28)}$$

Inserting (A7_25) into (A7_28), we get:

$$\dot{B}(t) = H\left[\frac{W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} + r(t)B_H(t)}{L(t)} - n\frac{B_H(t)}{L(t)}\right] \qquad \text{(A7\_29)}$$

Using (A7_26), equation (A7_29) yields:

$$\dot{B}(t) = H\left[\frac{W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} + r(t)\frac{L(t)}{H}B(t)}{L(t)} - n\frac{\frac{L(t)}{H}B(t)}{L(t)}\right] \qquad \text{(A7\_30)}$$

Thus, we arrive at a slightly modified version of equation (A7_24):

$$\dot{B}(t) = W(t) - C(t) + r(t)B(t) - nB(t) \qquad \text{(A7\_31)}$$

Because the size of the family is growing at rate n, this term negatively affects the increase in assets of one single member.

The No-Ponzi-Game condition for (A7_31) is as follows:

$$\lim_{t\to\infty} e^{-[R(t)-nt]}B(t) \geq 0 \qquad \text{(A7\_32)}$$

(A7_32) might be derived from (A7_25b) by the following steps:

$$\lim_{t\to\infty} e^{-R(t)}B_H(t) = \lim_{t\to\infty} e^{-R(t)}\frac{L(t)}{H}B(t) = \lim_{t\to\infty} e^{-R(t)}\frac{L(0)e^{nt}}{H}B(t) = \frac{L(0)}{H}\lim_{t\to\infty} e^{-R(t)}e^{nt}B(t) \geq 0$$
$$\text{(A7\_33)}$$

Since L(0)/H is surely positive, (A7_33) results in (A7_32).

Provided that (A7_25b) holds with equality due to monotonically increasing utility function, the intertemporal budget constraint implied by (A7_25) and (A7_25b) might be expressed as (see section B):

$$\int_0^\infty e^{-R(t)} C(t) \frac{L(t)}{H} dt = \frac{K(0)}{H} + \int_0^\infty e^{-R(t)} W(t) \frac{L(t)}{H} dt \qquad \text{(A7\_34)}$$

Equation (A7_34) simply states that the discounted flow of consumption of a representative household might not exceed (is equal to) the discounted flow of labour income plus the initial value of assets $B_H(0)=K(0)/H$. In this model, we assume that each household owns a proportional part of the initial stock of capital in the economy. (A7_34) is a continuous time version of the thoroughly discussed budget constraint (A5_12) from Appendix 5.

The objective of our household is to maximize lifetime utility function (A7_12) with respect to the flow constraint (A7_31) (or A7_25) or alternatively with respect to the intertemporal budget constraint (A7_34). Inserting equation (A7_31) in (A7_12) for C(t)=W(t)+r(t)B(t)-nB(t)-dB(t)/dt, and (A7_6) for the evolution of the labour force, our dynamic optimization problem is as follows:

$$\max U = \int_0^\infty e^{-\rho t} \frac{\{W(t) + [r(t) - n]B(t) - \dot{B}(t)\}^{1-\theta}}{1-\theta} \frac{L(0)e^{nt}}{H} dt \qquad \text{(A7\_35)}$$

There are two choice variables - the amount of assets at time t (i.e. B(t)) and the instantaneous change of assets at time t (i.e. dB(t)/dt). Solution of (A7_35) might be found with the help of the Euler equation (see Kamien, Schwartz:???):

$$\frac{\partial F(\cdot)}{\partial B(t)} = \frac{d \dfrac{\partial F(\cdot)}{\partial \dot{B}(t)}}{dt} \qquad \text{(A7\_36)}$$

where F(.) is the expression under integral in (A7_35). Thus:

$$\frac{\partial F(\cdot)}{\partial B(t)} = e^{-(\rho-n)t} C(t)^{-\theta} [r(t) - n] \frac{L(0)}{H} \qquad \text{(A7\_37)}$$

$$\frac{\partial F(\cdot)}{\partial \dot{B}(t)} = e^{-(\rho-n)t} C(t)^{-\theta} (-1) \frac{L(0)}{H} \qquad \text{(A7\_38)}$$

$$\frac{d \dfrac{\partial F(\cdot)}{\partial \dot{B}(t)}}{dt} = -\frac{L(0)}{H} \left[e^{-(\rho-n)t} C(t)^{-\theta} (-\rho + n) + e^{-(\rho-n)t} (-\theta) C(t)^{-\theta-1} \dot{C}(t) \right] \qquad \text{(A7\_39)}$$

(A7_36) gives us:

$$e^{-(\rho-n)t}C(t)^{-\theta}\left[r(t)-n\right]\frac{L(0)}{H}=-\frac{L(0)}{H}\left[e^{-(\rho-n)t}C(t)^{-\theta}(-\rho+n)+e^{-(\rho-n)t}(-\theta)C(t)^{-\theta-1}\dot{C}(t)\right]$$
(A7_40)

$$\left[r(t)-n\right]=-\left[(-\rho+n)+(-\theta)\frac{\dot{C}(t)}{C(t)}\right]$$ (A7_41)

The optimum growth rate of consumption is:

$$\frac{\dot{C}(t)}{C(t)}=\frac{r(t)-\rho}{\theta}$$ (A7_42)

As we can see, we arrived at a very simple consumption Euler equation known from previous sections. Furthermore, by realizing the fact that c(t)=C(t)/A(t), the growth rate of c(t) is:

$$\frac{\dot{c}(t)}{c(t)}=\frac{\dot{C}(t)}{C(t)}-\frac{\dot{A}(t)}{A(t)}=\frac{r(t)-\rho}{\theta}-g=\frac{r(t)-\rho-\theta g}{\theta}$$ (A7_43)

(A7_43) would be virtually the same as the social planner solution (A7_22), provided that r(t)=f´(k(t))-δ.

If we focus on the second part of the capital market, i.e. on firms, (A7_43) might be easily reconciled with (A7_22). The profit maximizing firm should, according to the neoclassical theory, equalize the marginal product of capital with the marginal cost of capital, which is equal, in a one good economy, to the sum of the real interest rate and depreciation rate. The proof of this statement runs as follows:

$$\pi_i=Y_i-\left[WL_i+(r+\delta)K_i\right]$$ (A7_44)

(A7_44) is the real profit function of a representative firm i. $Y_i = F(K_i,AL_i)$ stands for real revenues, the remaining part represents total real costs. Notice that real costs of capital are represented by real interest for capital rK and the amount of depreciation δK.

FOC in case of capital is:

$$\frac{\partial\pi}{\partial K_i}=\frac{\partial F}{\partial K_i}-(r+\delta)=0$$ (A7_45)

Thus:

$$\frac{\partial F(\cdot)}{\partial K_i}=MPK=r+\delta$$ (A7_46) **(A7_45)b**

At the same time, the production function is homogenous of degree one, hence its first derivative is homogenous of degree zero. This implies that f´(k)=MPK. The formal proof is simple:

$$MPK = \frac{\partial F(\cdot)}{\partial K} = \frac{\partial [ALf(k)]}{\partial K} = ALf'(k)\frac{1}{AL} = f'(k) \qquad \text{(A7\_47) } \textbf{(A7\_46)}$$

The real interest rate acts as the coordinating mechanism that will reconcile the behaviour of utility maximizing households (equation (A7_42)) and profit maximizing firms (A7_46). Using (A7_47) and (A7_46), equation (A7_43) for a decentralized economy might be written as (A7_22), which is the optimum solution for a social planner.

SS

Consumption per effective worker reaches its steady state (and consumption and, as we will see, other most important variables in the model (capital, output, real wage) their balanced growth path), if (A7_22) is equal to zero:

$$\dot{c}(t) = 0 \Leftrightarrow f'(k^*(t)) - \delta = \rho + \theta g \Leftrightarrow MPK^* - \delta = \rho + \theta g \qquad \text{(A7\_23) } \textbf{(A7\_47)}$$

Using (A7_43) for a decentralized economy or (A7_22) for a centrally planned economy, the steady state level of the natural rate of interest is as follows:

$$MPK - \delta = r^* = \rho + \theta g \qquad \text{(A7\_48)}$$

Since the marginal product of capital is an endogenous variable, it will always adjust to equal the right-hand side of this equation. Thus, the natural rate of interest in this neoclassical dynamic general equilibrium model is determined by the subjective discount rate (time preference in sense two) and the rate of technological progress, whose influence is modified by the curvature of the utility function.[61] At the same time, **(A7_47)** determines the steady state value of capital per effective worker k^*, which represents the only level of capital per effective worker, for which consumption per effective worker is constant. This value might be implicitly found using (A7_48):

$$f'(k^*(t)) - \delta = \rho + \theta g \qquad \text{(A7\_49)}$$

Even though there are two crucial endogenous variables, capital and consumption, the size of k^* does not depend on the level of consumption per effective worker (see the vertical line in Figure no.1_A7). To close the model, let us find the steady state value of consumption per effective worker. Using (A7_11), combinations of capital per effective worker and consumption per effective worker for which the capital per effective worker is constant might be written as:

$$\dot{k}(t) = 0 \Leftrightarrow c(t) = f(k(t)) - (n + g + \delta)k(t) \quad \text{(A7\_50)}$$

By substituting k^* from (A7_49), we obtain c^*:

$$c^* = f(k^*) - (n + g + \delta)k^* \qquad \text{(A7\_51)}$$

[61] It can be shown that if our modeling technique was slightly different, namely if we modeled a representative individual, who does not care about population growth, rather than a representative household which is concerned about the rate of its expansion, the natural real rate of interest at the steady state (i.e. in the dynamic general equilibrium) would also positively depend on the rate of population growth (see Section E).

c^* can be found at the intersection of the vertical line at point k^* and the concave curve that represents points from equation (A7_50) (See Figure no. 1_A7) Combination c^* and k^* then represents the steady state of this model.

Moreover, condition (A7_14) implies that:

$$\rho + \theta g > n + g \qquad \text{(A7\_52)}$$

The left hand side is equal to real interest rate at steady state and the right hand side to the growth rate of GDP. Hence, the economy in the Ramsey model is always dynamically efficient. To prove this, let us determine the growth rate of GDP in the steady state of k^*.

First, we can use the fact that:

$$\dot{y}(t) = f'(k(t))\dot{k}(t) \qquad \text{(A7\_53)}$$

At the steady state, k^* is constant, thus:

$$\dot{y}^* = f'(k^*)\dot{k}^* = 0 \qquad \text{(A7\_54)}$$

Furthermore, since y=Y/AL, the growth rate of Y at the steady state is:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\frac{d\left[y(t)A(t)L(t)\right]}{dt}}{y(t)A(t)L(t)} = \frac{\dot{y}(t)}{y(t)} + \frac{\dot{A}(t)}{A(t)} + \frac{\dot{L}(t)}{L(t)} = 0 + g + n \qquad \text{(A7\_55)}$$

The economy is considered as dynamically efficient, if it is not possible to raise consumption of some agents in a given period without reducing consumption of (the same or other) agents in some other periods. Technically it means that an increase in capital (due to higher saving, thus lower present consumption) will result in higher consumption in the future (in the new steady state) and conversely, a reduction in capital (due to lower saving, thus higher present consumption) will result in lower consumption in the future (in the new steady state). As a result, steady state consumption must positively respond to an increase in capital. Using (A7_51), (A7_55) and (A7_46) we can express this positive relationship as:

$$\frac{\partial c^*}{\partial k^*} > 0 \Leftrightarrow f'(k^*) - \delta > n + g \Leftrightarrow r^* > n + g \Leftrightarrow r^* > \frac{\dot{Y}(t)}{Y(t)} \qquad \text{(A7\_56)}$$

Thus, the condition for dynamic efficiency is that the real interest rate is greater than the growth rate of the economy.

Consumption is then maximized, if:

$$\frac{\partial c^*}{\partial k^*} = 0 \Leftrightarrow f'(k^*) - \delta = n + g \Leftrightarrow r^* = n + g \Leftrightarrow r^* = \frac{\dot{Y}(t)}{Y(t)} \qquad \text{(A7\_57)}$$

The level of capital that satisfies (A7_57) is known as the golden rule level of capital. However, capital at the steady state in the RCK model is always lower than golden rule due to condition (A7_52). k^* in the RCK model is sometimes called a modified golden rule.

At the end of this section, we can easily determine the optimum steady state growth rate of consumption. Using (A7_48) and (A7_42), it can be derived that on the balanced growth path it grows at the same rate as GDP per head, namely g:

$$\frac{\dot{C}(t)}{C(t)} = \frac{r^* - \rho}{\theta} = \frac{\rho + \theta g - \rho}{\theta} = g \qquad \text{(A7\_58)}$$

Section B. The intertemporal budget constraint in the infinite horizon continuous time model

In section A, it was stated that the flow constraint (A7_25) together with the No-Ponzi-Game condition (A7_25b) imply a continuous time version of the intertemporal budget constraint (A7_34), thoroughly discussed in the discrete time in Appendix 4. Let us stress that equation (A7_25) is a simple differential equation which might be rewritten as:

$$\dot{B}_H(t) - r(t)B_H(t) = W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} \qquad \text{(A7\_1B)}$$

Suppose that in the initial period, each representative household owns a proportional part of the total initial capital stock. This might be represented as:

$$B_H(0) = \frac{K(0)}{H} \qquad \text{(A7\_2B)}$$

Hence, (A7_2B) is the initial condition of the differential equation (A7_1B), solution of which might be derived with the help of the method of the variation of constants. Let us start with its homogenous representation:

$$\dot{B}_H(t) - r(t)B_H(t) = 0 \qquad \text{(A7\_3B)}$$

Solution of (A7_3B) runs as follows:

$$\frac{dB_H(t)}{dt} = r(t)B_H(t) \qquad \text{(A7\_4B)}$$

$$\int \frac{dB_H(t)}{B_H(t)} = \int r(t)dt \qquad \text{(A7\_5B)}$$

$$\ln B_H(t) = R(t) + d \qquad \text{(A7\_6B)}$$

(A7_6B) uses the fact that r(t) is the first derivative of R(t) (see the discussion below). Furthermore, d is an arbitrary constant of integration. (A7_6B) implies:

$$B_H(t) = D(t)e^{R(t)} \qquad \text{(A7\_7B)}$$

D(t) is simply exp(d), but in this method it is itself a function of time. Differentiation of (A7_6B) with respect to time yields:

$$\dot{B}_H(t) = \dot{D}(t)e^{R(t)} + D(t)e^{R(t)}r(t) \qquad \text{(A7\_8B)}$$

(A7_8B) uses the Leibnitz rule for differentiation of the integral with respect to the variable in the upper (or lower) limit, generally stated as (see Kamien,Schwartz ???):

$$\frac{dV(t)}{dt} = \frac{d\int_{A(t)}^{B(t)} f(\tau,t)d\tau}{dt} = f(B(t),t)B'(t) - f(A(t),t)A'(t) + \int_{A(t)}^{B(t)} \frac{\partial f(t,\tau)}{\partial t} d\tau \qquad \text{(A7\_8Bb)}$$

For a constant lower limit and a simple form we deal with in (A7_25c), (A7_8Bb) is modified to:

$$\frac{dV(t)}{dt} = \frac{d\int_{0}^{B(t)} f(\tau)d\tau}{dt} = f(B(t))B'(t) - 0 + 0 \qquad \text{(A7\_8Bc)}$$

Applying (A7_8Bc) to (A7_25c), we get:

$$\frac{dR(t)}{dt} = \frac{d\int_0^t r(\tau)d\tau}{dt} = r(\tau)\Big|_{\tau=t} \frac{dt}{dt} = r(t) \qquad \text{(A7\_8Bd)}$$

Thus, the time derivative of R(t) is simply r(t). By substituting (A7_8B) and (A7_7B) back to the original differential equation (A7_1B), we may write:

$$\dot{D}(t)e^{R(t)} + D(t)e^{R(t)}r(t) - r(t)D(t)e^{R(t)} = W(t)\frac{L(t)}{H} - C(t)\frac{L(t)}{H} \qquad \text{(A7\_9B)}$$

$$\dot{D}(t) = e^{-R(t)}W(t)\frac{L(t)}{H} - e^{-R(t)}C(t)\frac{L(t)}{H} \qquad \text{(A7\_10B)}$$

By integrating (A7_10B), we get:

$$D(t) = \int e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int e^{-R(t)}C(t)\frac{L(t)}{H}dt + F \qquad \text{(A7\_11B)}$$

F is an arbitrary constant of integration. Let us insert (A7_11B) into equation (A7_7B):

$$B_H(t)e^{-R(t)} = \int e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int e^{-R(t)}C(t)\frac{L(t)}{H}dt + F \qquad \text{(A7\_12B)}$$

At any time T, (A7_12B) can be written as:

$$B_H(T)e^{-R(T)} = \int_{t=0}^{T} e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int_{t=0}^{T} e^{-R(t)}C(t)\frac{L(t)}{H}dt + F \qquad \text{(A7\_13B)}$$

To determine the arbitrary constant F, we can use the initial condition (A7_2B) and the fact that R(0)=0.[62] Equation (A7_13B) for T=0 yields:

$$B_H(0)e^{-R(0)} = \int_{t=0}^{0} e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int_{t=0}^{0} e^{-R(t)}C(t)\frac{L(t)}{H}dt + F \qquad \text{(A7\_14B)}$$

$B_H(0) = F$ (A7_15B)

Using this fact and (A7_2B), (A7_13B) is simply:

$$B_H(T)e^{-R(T)} = \int_{t=0}^{T} e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int_{t=0}^{T} e^{-R(t)}C(t)\frac{L(t)}{H}dt + \frac{K(0)}{H} \qquad \text{(A7\_16B)}$$

On the other hand, if T goes to infinity, (A7_16B) yields:

$$\lim_{T\to\infty} B_H(T)e^{-R(T)} = \lim_{T\to\infty}\left\{\int_{t=0}^{T} e^{-R(t)}W(t)\frac{L(t)}{H}dt - \int_{0}^{T} e^{-R(t)}C(t)\frac{L(t)}{H}dt + \frac{K(0)}{H}\right\} \qquad \text{(A7\_17B)}$$

However, according to the No-Ponzi-Game condition (A7_25b), the left-hand side of (A7_17B) must be non-negative. Hence, (A7_17B) implies:

$$\int_{0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt \le \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}dt \qquad \text{(A7\_17C)}$$

(A7_17C) states that the present value of the flow of consumption in the infinite horizon may not exceed the present value of the flow of labour income plus the value of the initial assets.

Moreover, it would be suboptimal for the household not to completely exhaust all lifetime resources, since the lifetime utility has no bliss point. Thus, (A7_17C) is satisfied with equality leading to (A7_34).

Section C: Transforming the flow constraint in discrete time to continuous time:

In this section, we will show how the flow constraint in discrete time (A7_23) might be converted to a continuous time version (A7_24). Before that, however, we must slightly modify the discrete time equation, since our approach from Appendix 5 was not perfectly accurate.

Suppose that at the beginning of period zero, initial assets of our representative agent are represented by B_0. At the end of this period, assets (or debt) are increased or decreased by the difference between the flow of the labour income W_0 and the flow of consumption C_0 in this particular period. It should be stressed again that income and consumption are flow concepts,

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[62] This stems from the fact that $R(t) = \int_0^t r(\tau)d\tau$, thus $R(0) = \int_0^0 r(\tau)d\tau = 0$. Similar idea holds for the value of all integrals in (A7_13B).

whereas assets represent a stock concept. Hence, assets at the end of period zero should be denoted as B_1:

$$B_1 = W_0 - C_0 + B_0 \qquad \text{(A7\_1C)}$$

At the end of period zero (or at the beginning of period one), the interest is paid on the accumulated assets. This serves as an additional source for consumption and saving in the next period together with the labour income. Thus, the next period budget constraint might be represented as:

$$B_2 + C_1 = W_1 + B_1(1 + r_1) \qquad \text{(A7\_2C)}$$

(A7_2C) states that the assets at the end of period one, i.e. B_2, depend on the difference between the flow of labour income in this particular period, i.e. W_1, and the flow of consumption C_1. At the same time, assets from the previous period B_1 are increased by the interest r_1B_1.

This idea should hold in any time, equation (A7_2C) can be therefore generalized to:

$$B_{t+1} + C_t = W_t + B_t(1 + r_t) \qquad \text{(A7\_3C)}$$

This equation, in turn, may be rearranged to:

$$B_{t+1} - B_t = W_t - C_t + r_tB_t \qquad \text{(A7\_4C)}$$

In (A7_4C) the time difference between B_{t+1} and B_t is one period. Over that period, assets are either accumulated, partly reduced or remain the same. The eventual result depends on the variables on the right hand side of the equation. However, if the time period is halved, the accumulation will be also (roughly) halved, i.e. $B(t+1/2) – B(t) = ½(B_{t+1} - B_t)$. Obviously, halving was chosen arbitrarily, since any reduction in the given time period is possible. Thus, (A7_4C) might be expressed as:

$$B(t+s) - B(t) = s(W(t) - C(t) + r(t)B(t)) \qquad \text{(A7\_5C)}$$

s might be any number, e.g. one half. Let us divide (A7_5C) by s and assume that s approaches 0 in the limit. Equation (A7_5C) can be then written as:

$$\lim_{s \to 0} \frac{B(t+s) - B(t)}{s} = \lim_{s \to 0} \left[W(t) - C(t) + r(t)B(t) \right] \qquad \text{(A7\_6C)}$$

The left-hand side of (A7_2C) is simply the time derivative of B(t). Thus, we arrive at equation (A7_24):

$$\dot{B}(t) = W(t) - C(t) + r(t)B(t) \qquad \text{(A7\_24)}$$

The constraint (A7_24) simply states that the instantaneous change in assets depends on the difference between the instantaneous flow of labour income W(t) plus the instantaneous flow of interest r(t)B(t) minus the instantaneous flow of consumption C(t).

Section D: Behaviour of the economy around the steady state

In this section, we will analyse behaviour of the economy around its steady state. This will help us understand paths of consumption, capital and natural rate of interest after various shocks.

dk/dt can be approximated around the steady state k^*, c^* in the system of two differential equations (A7_11) - the law of motion of capital, and (A7_22) - the Euler equation indicating the optimum path of consumption, as follows:

$$\dot{k}(t) = \dot{k}(t)\Big|_{k(t)=k^*,c(t)=c^*} + \frac{\partial \dot{k}(t)}{\partial k(t)}\Big|_{k(t)=k^*,c(t)=c^*}(k(t)-k^*) + \frac{\partial \dot{k}(t)}{\partial c(t)}\Big|_{k(t)=k^*,c(t)=c^*}(c(t)-c^*)$$

(A7_1D)

In the discussion in Figure no. 45 in the main text, we consider constant population and technology and a sudden one-time increase in the level of technologies. Thus, (A7_11) might be written as:

$$\dot{k}(t) = Af(k(t)) - c(t) - \delta.k(t) \qquad \text{(A7\_2D)[63]}$$

Parameter A measures the particular level of technologies that is assumed to be constant. Neglecting time index for simplicity, (A7_1D) can be applied on (A7_2D) as follows:

$$\dot{k}(t) = 0 + \left[Af'(k^*) - \delta\right](k-k^*) + (-1)(c-c^*) \qquad \text{(A7\_3D)}$$

dc/dt is approximated around the steady state k^*, c^* in the same system as:

$$\dot{c}(t) = \dot{c}(t)\Big|_{k(t)=k^*,c(t)=c^*} + \frac{\partial \dot{c}(t)}{\partial k(t)}\Big|_{k(t)=k^*,c(t)=c^*}(k(t)-k^*) + \frac{\partial \dot{c}(t)}{\partial c(t)}\Big|_{k(t)=k^*,c(t)=c^*}(c(t)-c^*)$$

(A7_4D)

The Euler equation (A7_22) with constant technology might be written as:

$$\dot{c}(t) = \frac{Af'(k(t)) - \delta - \rho}{\theta}c(t) \qquad \text{(A7\_5D)}$$

Applying (A7_4D) on (A7_5D) and neglecting the time index, we get:

$$\dot{c} = 0 + \frac{Af''(k^*)}{\theta}c^*(k-k^*) + \frac{Af'(k^*) - \delta - \rho}{\theta}(c-c^*) \qquad \text{(A7\_6D)}$$

However, since $Af'(k^*)-\delta-\rho = 0$ at the steady state, (A7_6D) yields:

[63] Production function in the extensive form assumed in (A7_2D) is Y=AF(K,L), in the intensive form y=Af(k). Furthermore, there is no need to distinguish between consumption per worker C and consumption per effective worker c=C/A, since (labour-augmenting) technology is constant.

$$\dot{c} = 0 + \frac{Af''(k^*)}{\theta} c^* (k - k^*) \qquad \text{(A7\_7D)}$$

At the same time, using Af´(k*)-δ-ρ = 0 in (A7_3D), we get:

$$\dot{k} = 0 + \rho(k - k^*) + (-1)(c - c^*) \qquad \text{(A7\_8D)}$$

(A7_3D) and (A7_8D) constitute a system of two differential equations, representing an (linear) approximation of the economy around its steady state. To solve this system, let us differentiate (A7_8D) with respect to time:

$$\ddot{k} = \rho.\dot{k} - \dot{c} \qquad \text{(A7\_9D)}$$

Inserting (A7_7D) to (A7_9D) yields:

$$\ddot{k} = \rho.\dot{k} - \frac{Af''(k^*)}{\theta} c^* (k - k^*) \qquad \text{(A7\_10D)}$$

We may represent (A7_10D) as:

$$\ddot{k} - \rho.\dot{k} + \frac{Af''(k^*)c^*}{\theta} k = \frac{Af''(k^*)c^*}{\theta} k^* \qquad \text{(A7\_11D)}$$

Solution of this second-order differential equation can be found as follows:

$$\ddot{k} - \rho.\dot{k} + \frac{Af''(k^*)c^*}{\theta} k = 0 \qquad \text{(A7\_12D)}$$

(A7_12D) is the homogenous representation of (A7_11D). The characteristic equation is:

$$\lambda^2 - \rho.\lambda + \frac{Af''(k^*)c^*}{\theta} = 0 \qquad \text{(A7\_13D)}$$

The same equation would be obtained, if we designed a characteristic matrix of the system (A7_8D) and (A7_7D) and its determinant:

$$\det\begin{pmatrix} \rho - \lambda & -1 \\ Af''(k^*)c^* / \theta & 0 - \lambda \end{pmatrix} = \lambda^2 - \rho.\lambda + \frac{Af''(k^*)c^*}{\theta} \qquad \text{(A7\_14D)}$$

The two roots of (A7_13D) are:

$$\lambda_{1,2} = \frac{\rho \pm \sqrt{\rho^2 - 4\frac{Af''(k^*)c^*}{\theta}}}{2} \qquad \text{(A7\_15D)}$$

It should be perfectly clear that one root is greater than zero, $\lambda_1>0$. The second one is negative, $\lambda_2<0$, because $f''(k^*)$ is negative by assumption (production function is concave, marginal product is diminishing). Thus, the term under the square root is positive and greater than ρ. As a result, the solution has a saddle path property.

The particular solution k_p (a constant) of (A7_11D) is:

$$\ddot{k}_p - \rho.\dot{k}_p + \frac{Af''(k^*)c^*}{\theta}k_p = \frac{Af''(k^*)c^*}{\theta}k^* \qquad \text{(A7\_16D)}$$

$$k_p = k^* \qquad \text{(A7\_17D)}$$

Hence, the general solution of (A7_11D) is as follows:

$$k(t) = b_1 \exp(\lambda_1 t) + b_2 \exp(\lambda_2 t) + k^* \qquad \text{(A7\_18D)}$$

However, since $\lambda_1>0$, b_1 must be set equal to zero. λ_1 is associated with the unstable arm in Figure no. 1_A7. On the other hand, $\lambda_2<0$ is linked to the stable arm. Hence, from now on, λ_2 will be denoted simply as λ. The last step is to determine the arbitrary constant b_2. Using the initial condition $k(t=0) = k(0)$, (A7_18D) yields:

$$k(0) = 0\exp(\lambda_1 t) + b_2 \exp(\lambda_2 0) + k^* \qquad \text{(A7\_19D)}$$

$$b_2 = k(0) - k^* \qquad \text{(A7\_20D)}$$

Inserting (A7_20D) into (A7_18D) and putting $b_1=0$, we get:

$$k(t) = [k(0) - k^*].e^{\lambda t} + k^* \qquad \text{(A7\_21D)}$$

where

$$\lambda = \frac{\rho - \sqrt{\rho^2 - 4\frac{Af''(k^*)c^*}{\theta}}}{2} < 0 \qquad \text{(A7\_22D)}$$

As can be seen, the negativity of λ leads to the fact, that $k(t)$ gradually converges to the steady state k^*. Before expressing k^*, however, let us first find a similar convergence equation also for consumption.

From (A7_2D), we can write:

$$c(t) = Af(k(t)) - \delta.k(t) - \dot{k}(t) \qquad \text{(A7\_23D)}$$

Substituting (A7_21D) and its first time derivative into (A7_23D), we get:

$$c(t) = Af(k(t)) - \delta.k(t) - \lambda\left[k(0) - k^*\right]e^{\lambda t} \qquad \text{(A7\_24D)}$$

$$c(t) = Af(\left[k(0) - k^*\right]e^{\lambda t} + k^*) - \delta.\left\{\left[k(0) - k^*\right]e^{\lambda t} + k^*\right\} - \lambda\left[k(0) - k^*\right]e^{\lambda t} \qquad \text{(A7\_25D)}$$

$$c(t) = Af(\left[k(0) - k^*\right]e^{\lambda t} + k^*) - (\delta + \lambda).\left[k(0) - k^*\right]e^{\lambda t} - \delta.k^* \qquad \text{(A7\_26D)}$$

(A7_26D) might be used to determine the optimum consumption at time 0 for some given initial capital stock k(0):

$$c(0) = Af(\left[k(0) - k^*\right] + k^*) - (\delta + \lambda).\left[k(0) - k^*\right] - \delta.k^* \qquad \text{(A7\_27D)}$$

$$c(0) = Af(k(0)) - \delta.k(0) - \lambda.\left[k(0) - k^*\right] \qquad \text{(A7\_28D)}$$

Optimum c(0) along with k(0) are depicted in Figure no. 1_A7. Moreover, the equation of the saddle path, relating optimum c(t) – the control variable to k(t) – the state variable, known also as the policy function (Barro ???:105) might be determined by using (A7_24D) and (A7_21D):

$$c(t) = Af(k(t)) - \delta.k(t) - \lambda\left[k(t) - k^*\right] \qquad \text{(A7\_29D)}$$

$$c(t) = Af(k(t)) - (\delta + \lambda).k(t) + \lambda.k^* \qquad \text{(A7\_30D)}$$

As we will see, λ is lower (in negative value) with higher θ. Thus, saddle path is closer to the capital locus dk/dt=0 with higher θ.

Finally, let us determine the steady state value of capital, consumption and the saving rate. To solve these steady state values, however, we need to assume a specific form of the production function. Thus, let us consider a simple Cobb-Douglas form:

$$Y(t) = AK(t)^{\alpha} L^{1-\alpha} \qquad \text{(A7\_31D)}$$

As we assumed before, A and L are constant. The intensive form might be obtained by dividing (A7_31D) by L:

$$y(t) = Ak(t)^{\alpha} \qquad \text{(A7\_32D)}$$

where y=Y/L and k=K/L. To find the steady state value of k^*, let us use (A7_47) for g=0:

$$A\alpha k^{\alpha-1} - \delta = \rho \qquad \text{(A7\_33D)}$$

(A7_33D) states that in the steady state, the natural rate of interest $r^* = MPK - \delta$ is equal to the subjective discount rate ρ. The steady state of k(t) is thus:

$$k^* = \left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_34D)}$$

Steady state level of consumption is (using A7_51 for n=g=0):

$$c^* = A\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_35D)}$$

We can compare this level with the golden rule that is derived from (A7_57):[64]

$$A\alpha k^{\alpha-1} = \delta \qquad \text{(A7\_36D)}$$

The golden rule level of capital is:

$$k_{GR} = \left(\frac{A\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_37D)}$$

Obviously, k^* is lower than k_{GR} as long as the subjective discount rate ρ is positive. Hence c^* is also lower than c_{GR}:

$$c_{GR} = A\left(\frac{A\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{A\alpha}{\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_38 D)}$$

The optimum saving rate might be also determined from (A7_11). For n=g=0, we get:

$$s^* f(k^*) = \delta . k^* \qquad \text{(A7\_39 D)}$$

Which yields:

$$s^* = \delta . \frac{\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}}{A\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}} \qquad \text{(A7\_40 D)}$$

$$s^* = \frac{\delta}{A} . \left(\frac{A\alpha}{\rho+\delta}\right) \qquad \text{(A7\_41 D)}$$

[64] As can be seen, in our case the condition for the golden rule is $MPK = \delta$.

$$s^* = \frac{\delta\alpha}{\rho+\delta} \qquad \text{(A7\_42 D)}$$

As can be seen, optimum saving rate in the steady state is positively related to the depreciation rate and negatively to the subjective discount rate. Surprisingly, it does not depend on the level of technologies A.

Furthermore, golden rule level of saving is simply:

$$s_{GR} = \frac{\delta k_{GR}}{y_{GR}} \qquad \text{(A7\_43 D)}$$

$$s_{GR} = \frac{\delta.\left(\frac{A\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}}{A\left(\frac{A\alpha}{\delta}\right)^{\frac{\alpha}{1-\alpha}}} \qquad \text{(A7\_44 D)}$$

$$s_{GR} = \frac{\delta.\left(\frac{A\alpha}{\delta}\right)}{A} \qquad \text{(A7\_45 D)}$$

$$s_{GR} = \alpha \qquad \text{(A7\_46 D)}$$

As can be seen, optimum saving rate is lower than the golden rule. People are too impatient ($\rho>0$) to save such a large part of their incomes. Interestingly enough, if we neglect also depreciation rate (i.e. $\delta=0$), the optimum saving rate in the steady state is zero (see A7_42 D). A picture of such an economy that is converging to a seemingly peculiar steady state is given in Figure no. 0a_A7. [65] At first glance, it might seem optimal to permanently accumulate capital, which will result in ever increasing income and consumption. However, impatient households will eventually choose zero saving (see A7_42 D and Figure no. 0b_A7) and some definite level of consumption (A7_47 D):[66]

$$c^* = A\left(\frac{A\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} \qquad \text{(A7\_47 D)}$$

And finally, k^* and c^* might be used to determine the speed of convergence $-\lambda$ in (A7_22D). Before that, however, we need to determine $f''(k^*)$. Thus, using (A7_32D) we get:

$$f''(k) = A\alpha(\alpha-1)k^{\alpha-2} \qquad \text{(A7\_48 D)}$$

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[65] As can be seen in Figure no. 0a_A7, there is no peak point on the capital locus dk/dt=0, since its equation (see equation A7_11 for $n=g=\delta=0$) implies $c(t)=Af(k(t))$ which in this case expands beyond all bounds with higher k.

[66] c_{sub} is not optimal, since the interest rate for such a state would be lower than the subjective discount rate. This would lead the consumer to choose much larger consumption (larger than income, hence consumers would directly consume capital) and a decreasing time shape of consumption. The economic reason is that consumers would be too impatient to keep positive saving for such a low interest rate.

At the steady state (see A7_34D), equation (A7_48 D) yields:

$$f''(k^*) = A\alpha(\alpha-1)\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha-2}{1-\alpha}} \qquad \text{(A7\_49D)}$$

Expression f´´(k*)c* in (A7_22D) is thus:

$$f''(k^*)c^* = A\alpha(\alpha-1)\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha-2}{1-\alpha}}\left[A\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}} - \delta\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}\right] =$$

$$= A^2\alpha(\alpha-1)\left(\frac{A\alpha}{\rho+\delta}\right)^{-2} - \delta A\alpha(\alpha-1)\left(\frac{A\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} = (\alpha-1)\frac{(\rho+\delta)^2}{\alpha} - \delta(\alpha-1)(\rho+\delta) =$$

$$= (\alpha-1)(\rho+\delta)(\frac{\rho+\delta}{\alpha} - \delta)$$

(A7_50D)

Hence, λ in (A7_22D) might be expressed as:

$$\lambda = \frac{\rho - \sqrt{\rho^2 - \frac{4A}{\theta}(\alpha-1)(\rho+\delta)(\frac{\rho+\delta}{\alpha} - \delta)}}{2} < 0 \qquad \text{(A7\_51D)}$$

$$\lambda = \frac{\rho - \sqrt{\rho^2 + \frac{4A}{\theta}\frac{(1-\alpha)}{\alpha}(\rho+\delta)(\rho+\delta-\alpha\delta)}}{2} < 0 \qquad \text{(A7\_52D)}$$

As can be seen, the speed of convergence -λ depends negatively on parameter θ. Hence the lower the elasticity of substitution (high θ), the lower the pace at which the economy moves towards its steady state. The reason is that with high θ, the saving function is rather inelastic. On the other hand, the speed of convergence -λ depends positively on the subjective discount rate ρ and the level of technologies A. Thus, higher impatience leads to faster convergence. The reason is that ρ does not affect the slope of the saving function but rather its position in the r-s space.

So far, we have developed enough tools to simulate the behaviour of the economy in response to various shocks. Let us first consider a (permanent) one-time increase in the level of technologies from A_1 to A_2. The evolution of the real interest rate is depicted in Figure no. 44 in the main text. In Figure no. 1_A7, we plot the impact of an increase in A in the Ramsey model and in the production part of the economy represented by the Solow model (Figure no. 2_A7).

As can be seen in Figure no.1_A7, the eventual steady state capital and consumption per worker are greater than in the initial steady state (see A7_47D and A7_34D). The same conclusion obviously holds for the output per worker (see Figure no. 2_A7). However, the steady state natural rate of interest (see A7_33D) is not affected by greater A, since it is determined solely by the time preference (in sense two), i.e. by the subjective discount rate ρ. The steady state optimum saving rate is not affected by the level of technologies either (see A7_42 D).[67]

More interesting might be the transition period when the economy moves from the old to the new steady state. As can be seen in Figure no. 45 in the main text, the real natural rate of interest suddenly increases after the rise in the level of technologies. However, it gradually falls to the original level that is solely determined by the time preference (in sense two).

Another interesting question is the behaviour of the optimum consumption immediately after the shock. Figure no. 1_A7 indicates that the optimum response of consumption depends on the shape of the saddle path. Low elasticity of substitution (high θ) is associated with the saddle path that is closer to the capital locus dk/dt=0. The reason is the strong preference for consumption smoothing resulting in a relatively steep saving curve (or even saving curve with negative slope).[68] Hence, an increase in the demand for capital and the investment demand leading to higher interest rate is followed in this case by a relatively modest response on the part of saving. Since higher level of technologies raises initial income and the impact on saving is relatively low, present consumption rises.

On the other hand, high intertemporal elasticity of substitution (low θ) implies a relatively flat saving curve resulting in a considerable increase in saving after the rise in the interest rate. Thus, present optimum consumption drops after the increase in the level of technologies. The economic reason is the low preference for consumption smoothing leading to significant consumption growth over time, once the real interest rate exceeds subjective discount rate (see Euler equation A7_42). Hence, to reach this rapid growth in consumption present consumption must fall.

Furthermore, the rapidity of convergence of the key variables is also affected by parameter θ. Simulations in figures below clearly demonstrate that higher θ is associated with less rapid convergence (Figure no. 3_A7 and 4_A7). Even more, these simulations clearly verify conclusions discussed above. Let us stress again a considerable increase in the optimum saving rate if θ is low compared with high θ (Figure no. 6_A7).

[67] The consumption locus in Figure no. 1_A7 shifts to the right, because the term A increased in equation (A7_34D). Furthermore, according to (A7_37D) the golden rule level of capital is higher than before along with the golden rule level of consumption (see A7_38 D). Hence, the capital locus expands, as can be seen in this figure as well.

[68] It can be shown that if θ exceeds 9.2, initial saving decreases. However, this does not mean that the saving curve is downward sloping for this particular value. There are two simultaneously operating phenomena. First, increase in A raises income and hence saving curve shifts to the right. And second, the eventual impact on the amount of saving also depends on the shape of the saving curve, which is determined by θ. Yet, these two phenomena cannot be distinguished in our simulations. Thus, saving curve becomes downward sloping for much lower values of θ than 9.2. See section F that analyses the role of θ in determining the impact of the real interest rate on present consumption. Furthermore, it can be shown that present consumption is not affected after the shock, if $\theta = 0.38$. Again, this does not mean that the saving curve is vertical, as one would suggest for invariant income, or horizontal as one may think if income moves with higher technology. The saving curve rather shifts to the right due to higher income. At the same time, it is very flat. Hence, an increase in income accompanied by a considerable increase in saving leads to zero change in present consumption for this very low value of θ.

Higher saving rate resulting from greater elasticity of substitution also leads to more rapid growth in output in periods that follow after the shock (see Figure no. 7_A7).[69] Nevertheless, this rapid growth gradually dies out as the growth rate returns to its initial level (which is zero in this case). As can be seen, in more remote periods growth is lower in the economy populated by consumers with higher intertemporal elasticity of substitution. This specific behaviour of the growth rate is a direct consequence of more rapid convergence of this particular economy. Relatively fast growth in the initial periods is followed by a sluggish growth in GDP in the more remote future.

The evolution of the real interest rate is simulated in Figure no. 8_A7. As can be seen, the initial increase after the technological shock is the same for all values of θ. Nevertheless, the decline to the steady state level is sharper in the economy with higher elasticity of substitution, since the accumulation of capital is faster in this case.

Figures no. 7_A7 and 8_A7 also indicate that (apart from the period of the shock)[70] the real interest rate is always greater than the growth rate of GDP. In other words, the economy is dynamically efficient also in the transition period between the two steady states.

Furthermore, keeping money and velocity constant the inflation rate is simply the inverse to the growth rate of real output. Since the real interest rate is greater than the growth rate in output (and hence the rate of price deflation), nominal interest rate is always positive.[71] As can be seen in Figure no. 9_A7, the nominal interest rate approaches its steady state level from below, if the elasticity of substitution is high (low θ). The reason is a rapid growth in GDP and (for constant money) significant price deflation.

We may conclude that even though the real interest rate is surely higher for all levels of θ, nominal interest rate might either increase or decrease in the periods after the shock. According to our analysis, initial increase in the nominal interest rate is associated with low intertemporal elasticity of substitution (high θ) due to the slow growth in GDP and the resulting modest price deflation. On the other hand, if the elasticity of substitution is high (low θ), the nominal interest rate declines owing to rapid price deflation that reflects fast economic growth.

We can also examine the role of the subjective discount rate after the increase in the level of technologies. However, since this parameter affects especially the position of the saving curve rather than its slope, we do not obtain any interesting observations. ρ affects mainly levels of the key variables, hence its role in the convergence process is not as interesting as in the case of θ. Yet, there is some role of the subjective discount rate in the speed of convergence. As was already said, lower ρ is associated with more slowly convergence as is depicted in Figures no. 10_A7 and 11_A7.

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[69] Before any accumulation of capital starts, i.e. in the period of shock, output rises by 10 % due to an increase in the level of technologies by the same magnitude.

[70] It should be stressed that the time unit in our figures is one year. However, since our model is continuous, the growth rate of 10 % is valid not in the entire year, but rather in the given instantaneous moment of the shock. Immediately after this shock, the growth rate falls below the real interest rate. See Figure no. 7_A7b for the dynamics of the growth rate per worker within the first year, where the time interval between subsequent periods is much shorter.

[71] In this model, we assume perfect foresight hence expected inflation is always equal to actual inflation. However, the shock itself is unexpected. Thus, the actual inflation rate cannot be reflected in the expected inflation rate at the moment of the shock and, as a result, in the nominal interest rate. It seems to be more appropriate to assume that the nominal interest rate at the time of shock is equal to the previous steady state level, so we can focus on the behaviour of the nominal interest rate in periods that follow the shock. Furthermore, this disturbing fact (which also creates a difference between the natural real rate of interest depicted in Figure no. 8_A7 and the ex post real interest rate) is valid only at one instantaneous moment of the shock, not in the whole year.

More interesting is the behaviour of the economy if the subjective discount rate changes itself. Let us consider a fall in the subjective discount rate from 5 % to 4 %. The graphical representation of this change in the RCK model is provided in Figure no. 43 in the main text. The static representation of lower time preference (in sense two) is represented in Figure no. 22 in the main text, if we move from point E2 to point E1. As can be seen, this shock will shift the entire saving curve and the natural rate of interest falls.

However, in the dynamic model presented here the subsequent dynamics of the economy is much more complicated than suggested by a simple static model. An increase in saving and the movement along the investment curve results in the fact that more capital is being accumulated. Thus, next period output is greater along with the next period income. As a result, saving curve shifts outwards as people earn greater income. This will further decrease the natural rate of interest and increase the amount of invested capital. Hence, more capital might be accumulated anew. Consequently, income will rise along with saving. This is the source for the new accumulation of capital in the next round.

The question is whether the drop in time preference can provoke eternal growth in income and a never-ending decrease in the natural rate of interest. As is obvious, the static model is rather inadequate to account for this complicated dynamics. In the first place, it does not reflect additional shifts in the saving curve that result from increasing income. In the second place, it cannot answer the question whether the saving curve is being indefinitely moved to the right.

It is the RCK model presented here that may provide us with the fundamental answers. It clearly demonstrates that the impact of lower subjective discount rate gradually dies out. The increase in income is not eternal due to the diminishing marginal productivity of capital and the impatience of people. Greater capital is subjected to greater depreciation, hence eternal accumulation is not possible under the diminishing marginal productivity (see Knight ???). Nonetheless, even if the depreciation rate was zero, positive (though lower) subjective rate of discount would act as a break for further accumulation of capital. The point of optimality in dynamic equilibrium c*, k* in Figure no. 0b_A7 would just move to the right (see Figure no. 0c A7 and d_A7). Obviously, there would be a sudden decrease in present consumption after the fall in ρ. Yet, even with no depreciation of capital, diminishing marginal productivity of capital and the resulting decrease in the real interest rate together with the presence of the positive subjective discount rate would bring the process of capital accumulation to a halt.

The set of figures below shows the evolution of the key variables after the fall in the subjective discount rate (and positive depreciation rate). They clearly demonstrate the transitory impact of the fall in the subjective discount rate. At the same time, the role of the intertemporal elasticity of substitution (θ) is also presented.

Figure no. 12_A7 clearly shows that lower θ is associated with faster convergence. The economic reason is the significant drop of present consumption (see Figure no. 13_A7). As was said before, higher elasticity of substitution is also associated with faster growth of GDP in the subsequent periods after the shock and lower growth in more remote future (Figure no. 14_A7). The economic reason is the considerable increase in the saving rate (see Figure no. 15_A7).

This figure also displays that the eventual level of the saving rate is greater for all values of θ compared with the initial level (see equation A7_42 D).[72] This value is also independent of θ. However, lower elasticity of substitution leads to less abrupt increase in saving after the shock due to the high preference for the consumption smoothing. As can be seen in the figure, the

[72] However, if the depreciation of capital is zero (δ=0), the eventual saving rate is the same as before the shock, namely zero (See Figure no. 0d_A7)

saving rate gradually falls to the new steady state level. Nevertheless, it can be shown that the saving rate in the transition period might approach the new steady state level from below, if the elasticity of substitution is low enough (Barro:???:109; 135-137).[73]

And finally, even though present consumption falls, the eventual level is greater than in the initial steady state. This is a direct consequence of the dynamic efficiency of this economy. As can be seen in Figures no. 16_A7 and 14_A7, real interest rate is greater than the growth rate of output even in the transition period. Thus, nominal interest rate is also positive, although there might be a temporary period of price deflation (for constant money and velocity). As can be seen in Figure no. 17_A7, the drop in the nominal interest rate after the shock is largest for low θ, reflecting not only rapid fall in the real interest rate but also a relatively higher price deflation resulting from a faster growth in output.

In the next section, we relax the assumption of constant population and technology. Yet, we will assume that all changes in technology will reflect only changes in the growth rate in labour-augmenting technological progress g. The effects of this change are also discussed in the main text.

Equations of approximations of the economy around its steady state (A7_1D) and (A7_4D) are the same as before. Compared with the previous case, the law of motion of capital is (see A7_11):

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t) \qquad \text{(A7\_53D)}$$

The Euler equation might be represented as (see A7_22):

$$\dot{c}(t) = \frac{f'(k(t)) - \delta - \rho - \theta g}{\theta} c(t) \qquad \text{(A7\_54D)}$$

Thus, the linear approximation of (A7_53D) is as follows:

$$\dot{k}(t) = 0 + \left[f'(k^*) - (n + g + \delta)\right](k - k^*) + (-1)(c - c^*) \qquad \text{(A7\_55D)}$$

The linear approximation of (A7_54D) gives us:

$$\dot{c} = 0 + \frac{f''(k^*)}{\theta} c^*(k - k^*) + \frac{f'(k^*) - \delta - \rho - \theta g}{\theta}(c - c^*) \qquad \text{(A7\_56D)}$$

Since the real interest rate at the steady state is $r^* = \rho + \theta g$ and because the optimum of profit maximizing firms requires $r = f'(k) - \delta$, (A7_55D) can be written as:

$$\dot{k}(t) = 0 + \left[\rho + \theta g - (n + g)\right](k - k^*) - (c - c^*) \qquad \text{(A7\_57D)}$$

Condition (A7_14) implies that the term in the brackets in (A7_57D) is positive. Let us denote this term as $\beta \equiv \rho - n - (1-\theta)g > 0$. Thus:

$$\dot{k}(t) = \beta(k - k^*) - (c - c^*) \qquad \text{(A7\_58D)}$$

[73] This θ is such that $1/\theta < s^*$, which, according to equation (A7_42 D), means $\theta = (\rho+\delta)/(\alpha\delta)$. By inserting parameters from our simulation, we get the critical value: θ=7.

Furthermore, condition of the steady state (A7_47) implies that the second term in (A7_56D) is zero:

$$\dot{c} = \frac{f''(k^*)}{\theta} c^* (k - k^*) \qquad \text{(A7\_59D)}$$

By comparing the system of equations (A7_58D) and (A7_59D) with (A7_7D) and (A7_8D), we can see that they are almost the same. The only difference is the presence of β rather than ρ in equation (A7_58D) and the absence of A in (A7_59D). Hence, applying the same methods as before, we get:

$$k(t) = [k(0) - k^*].e^{\lambda t} + k^* \qquad \text{(A7\_60D)}$$

where:

$$\lambda = \frac{\beta - \sqrt{\beta^2 - 4\dfrac{f''(k^*)c^*}{\theta}}}{2} < 0 \qquad \text{(A7\_61D)}$$

The solution for the motion of consumption can be obtained, if we use (A7_53D), (A7_60D) and the first time derivative of (A7_60D):

$$c(t) = f(k(t)) - (n + g + \delta).k(t) - \lambda.[k(0) - k^*].e^{\lambda t} \qquad \text{(A7\_62D)}$$

$$c(t) = f([k(0) - k^*].e^{\lambda t} + k^*) - (n + g + \delta).\{[k(0) - k^*].e^{\lambda t} + k^*\} - \lambda[k(0) - k^*].e^{\lambda t} \qquad \text{(A7\_63D)}$$

$$c(t) = f([k(0) - k^*].e^{\lambda t} + k^*) - (n + g + \delta + \lambda).[k(0) - k^*].e^{\lambda t} - (n + g + \delta).k^* \qquad \text{(A7\_64D)}$$

Optimum initial consumption is:

$$c(0) = f([k(0) - k^*] + k^*) - (n + g + \delta + \lambda).[k(0) - k^*] - (n + g + \delta).k^* \qquad \text{(A7\_65D)}$$

$$c(0) = f(k(0)) - (n + g + \delta).k(0) - \lambda.[k(0) - k^*] \qquad \text{(A7\_66D)}$$

The equation of the saddle path in this case is (using A7_60D and A7_62D):

$$c(t) = f(k(t)) - (n + g + \delta).k(t) - \lambda.[k(t) - k^*] \qquad \text{(A7\_67D)}$$

$$c(t) = f(k(t)) - (n + g + \delta + \lambda).k(t) + \lambda.k^* \qquad \text{(A7\_68D)}$$

The steady state values of capital (per effective worker), consumption (per effective worker) and saving rate might be determined by the same procedure as before. The specific form of the production function will be Cobb-Douglas again. However, now we assume that the population is growing over time and labour-augmenting technological progress is growing as well. Thus, the production function is:

$$Y(t) = K(t)^{\alpha}\left[A(t)L(t)\right]^{1-\alpha} \qquad \text{(A7\_69D)}$$

The intensive form of (A7_69D) can be derived by dividing the whole expression by AL:

$$y(t) = k(t)^{\alpha} \qquad \text{(A7\_70D)}$$

where y=Y/(AL) and k=K/(AL). To find the steady state value of k^*, let us use (A7_47) again:

$$\alpha k^{\alpha-1} - \delta = \rho + \theta.g \qquad \text{(A7\_71D)}$$

The steady state of k(t) is thus:

$$k^* = \left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_72D)}$$

The steady state level of consumption is (using A7_51):

$$c^* = \left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha}{1-\alpha}} - (n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_73D)}$$

Again, we can compare this level with the golden rule that is derived from (A7_57):

$$\alpha k^{\alpha-1} = n+g+\delta \qquad \text{(A7\_74D)}$$

The golden rule level of capital is:

$$k_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_75D)}$$

Obviously, k^* is lower than k_{GR} as long as $\rho+\delta+\theta g > n+g+\delta$. However, this implies that $\rho-n-(1-\theta)g>0$. This condition was arrived at many times before. Hence, c* is always lower than c_{GR}:

$$c_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} - (n+g+\delta)\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_76D)}$$

Let us determine the optimum saving rate at the steady state. Using (A7_11), we get:

$$s^* f(k^*) = (n+g+\delta).k^* \qquad \text{(A7\_77D)}$$

This yields (see A7_72D for k^* and A7_70D for the resulting $y^*=f(k^*)$):

$$s^* = (n+g+\delta)\frac{\left(\dfrac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{1}{1-\alpha}}}{\left(\dfrac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha}{1-\alpha}}} \qquad \text{(A7\_78D)}$$

$$s^* = (n+g+\delta)\frac{\alpha}{\rho+\delta+\theta.g} \qquad \text{(A7\_79D)}$$

As before, the optimum saving rate in the steady state is negatively related to the subjective discount rate. Moreover, it positively depends on the rate of population growth. The reason is that the dynastic family is concerned about the well-being of its offspring. Thus, any increase in the rate of the expansion of the household leads to an increase in saving in the effort to secure the optimum growth rate of consumption (see A7_42) for each of its member. More on this will be said in section E.

The effect of δ on s^* might be determined as follows:

$$\frac{\partial s^*}{\partial \delta} = \frac{\alpha(\rho+\delta+\theta.g) - \alpha(n+g+\delta)}{(\rho+\delta+\theta.g)^2} \qquad \text{(A7\_80D)}$$

(A7_80D) is definitely positive due to condition (A7_14). Thus, optimum saving in the steady state always increases with higher depreciation rate.

The effect of the growth rate of technological progress on s^* might be obtained by a similar procedure:

$$\frac{\partial s^*}{\partial g} = \frac{\alpha(\rho+\delta+\theta.g) - \alpha(n+g+\delta)\theta}{(\rho+\delta+\theta.g)^2} \qquad \text{(A7\_81D)}$$

(A7_81D) is positive if:

$$(\rho+\delta+\theta.g) > (n+g+\delta)\theta \qquad \text{(A7\_82D)}$$

This yields:

$$\rho > \theta.n + (\theta-1)\delta \qquad \text{(A7\_83D)}$$

Hence, faster technological progress will very likely increase the steady state optimum saving the greater the elasticity of substitution is (lower θ).

The golden rule level of saving is equal to α even in this case. This outcome can be easily proved generally. Due to condition (A7_14), s^* is lower than s_{GR} (compare A7_79D and A7_87D):

$$s_{GR}=\frac{(n+g+\delta)k_{GR}}{y_{GR}} \qquad \text{(A7\_84D)}$$

$$s_{GR}=\frac{(n+g+\delta)\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}} \qquad \text{(A7\_85D)}$$

$$s_{GR}=(n+g+\delta)\frac{\alpha}{n+g+\delta} \qquad \text{(A7\_86D)}$$

$$s_{GR}=\alpha \qquad \text{(A7\_87D)}$$

And finally, $k^*$and c^* might be used to determine the speed of convergence -λ in (A7_61D). Before that, however, we need to determine $f''(k^*)$. Thus, using (A7_70D) we get:

$$f''(k)=\alpha(\alpha-1)k^{\alpha-2} \qquad \text{(A7\_88D)}$$

At the steady state (see A7_72D), equation (A7_88D) yields:

$$f''(k^*)=\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha-2}{1-\alpha}} \qquad \text{(A7\_89 D)}$$

Expression $f''(k^*)c^*$ in (A7_61D) is thus:

$$f''(k^*)c^*=\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha-2}{1-\alpha}}\left[\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha}{1-\alpha}}-(n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{1}{1-\alpha}}\right]=$$

$$=\alpha(\alpha-1)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{-2}-\alpha(\alpha-1)(n+g+\delta)\left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{\alpha-1}{1-\alpha}}=$$

$$= (\alpha - 1)\frac{(\rho + \delta + \theta.g)^2}{\alpha} - (\alpha - 1)(n + g + \delta)(\rho + \delta + \theta.g) =$$

$$= (\alpha - 1)(\rho + \delta + \theta.g)\left[\frac{\rho + \delta + \theta.g}{\alpha} - (n + g + \delta)\right] \qquad \text{(A7\_90 D)}$$

Hence, λ in (A7_61D) can be expressed as:

$$\lambda = \frac{\beta - \sqrt{\beta^2 - 4\frac{(\alpha - 1)(\rho + \delta + \theta.g)}{\theta}\left[\frac{\rho + \delta + \theta.g}{\alpha} - (n + g + \delta)\right]}}{2} < 0 \qquad \text{(A7\_91 D)}$$

$$\lambda = \frac{\beta - \sqrt{\beta^2 + 4\frac{1 - \alpha}{\alpha}\frac{\rho + \delta + \theta.g}{\theta}\left[\rho + \delta + \theta.g - \alpha(n + g + \delta)\right]}}{2} < 0 \qquad \text{(A7\_92 D)}$$

As before, the speed of convergence is positively related to ρ and negatively related to θ.

Let us now simulate the behaviour of the economy after a sudden increase in the growth rate of technological progress g. In Figure no. 44 in the main text we show that the real natural rate of interest gradually increases after the rise in g. Here, we will discuss this shock in a more detail. We focus on the role of the preference for consumption smoothing (parameter θ) that reflects curvature of the intertemporal indifference curves.

First, Figure no. 18_A7 clearly shows that the elasticity of substitution affects the level of the optimum steady state saving rate, when the positive technological progress is present in the model. According to equation (A7_79D), the higher the elasticity of substitution (lower θ) the higher the steady state saving rate is. This can be explained by a flat saving curve associated with low θ and expanding investment curve fed by positive technological progress.

Second, the optimum response of saving to an increase in g also depends on θ. As can be seen in Figure no. 18_A7, lower elasticity of substitution (high θ) is associated with a sharp drop in the saving rate after the shock. The economic reason is the strong preference for consumption smoothing. Increase in g guarantees higher growth in the (future) income endowment. Thus, this higher future income our consumer shifts closer to the present via reduced saving.[74]

Third, whether the eventual steady state saving rate is lower or higher than the initial one also depends on the elasticity of substitution (see equations from A7_81D to A7_83D). As can be seen in Figure no. 18_A7, higher preference for consumption smoothing results in lower eventual steady state saving rate, even though the saving rate in the transition period gradually increases from very low levels observed immediately after the shock.

The behaviour of other fundamental variables then depends on the response of saving discussed above. Figure no. 19_A7 indicates that the most rapid growth in GDP per worker in

[74] On the other hand, very low θ (namely 0.3 for our set of parameters) leads to an increase in the saving rate immediately after the shock. In this case, the consumer easily shifts consumption over time and responds to gradually increasing real interest rate by reducing the present consumption.

the initial periods is triggered in the case of high elasticity of substitution. Thus, convergence is the fastest for low θ. However, the eventual growth rate in GDP per worker is dictated solely by g. Thus, the new steady state (BGP) growth will be higher regardless of the size of θ.

The response of saving is mirrored in the optimum behaviour of consumption. Figure no. 20_A7 reports the optimum growth rate of consumption per worker. As can be seen, high elasticity of substitution is associated with only a modest increase in the present consumption after the shock, which enables more rapid growth in consumption and faster convergence to the new steady state growth rate in the periods afterwards.

Furthermore, the elasticity of substitution critically affects the behaviour of the real natural rate of interest not only in the transition period, but also its level in the new steady state. Figure no. 21_A7 displays the evolution of the real interest rate for various θ. The most sluggish convergence is seen in case of high θ. At the same time, the eventual increase in the real rate of interest is the largest in this case as well (see equation A7_48).

Comparing Figure no. 21_A7 and 19_A7, the economy is always dynamically efficient, since the real interest rate always exceeds the growth rate in GDP. This conclusion is reflected (for constant money and velocity) in the evolution of the nominal interest rate (see Figure no. 22_A7). Yet, a very interesting observation might be found in this figure. For θ=1 (i.e. logarithmic utility function discussed many times in the previous text), the eventual nominal interest rate is not affected by higher technological progress, because higher real natural rate of interest is perfectly offset by more rapid price deflation that is implied by faster growth in GDP. As was discussed in a great detail in the previous sections, the nominal interest rate is not affected by productivity only in this very specific case.[75] Thus, the pure time preference approach is correct as regards the un-importance of productivity even in this complicated dynamic model provided that its theoretical basis is the interest on money (and not the intertemporal exchange of real goods) and the utility function is logarithmic. However, even in this very convenient environment for the Austrian PTPT, interest on money is affected by productivity in the transition period. At the same time, the period for which the nominal interest rate is different from the pure time preference ρ is rather long.

It should be stressed that high elasticity of substitution (θ<1) is associated with a drop in the nominal interest rate after the shock and also with a fall in its steady state level. The reason is that the real interest rate does not grow enough to offset higher price deflation resulting from more rapid economic growth. On the other hand, low elasticity of substitution (θ>1) leads to a negligible decrease in the nominal rate (it can even increase for very high θ) after the shock and, as can be seen in the figure, to an increase in the eventual nominal rate of interest. The reason is obviously a large increase in the natural steady state real rate of interest after the rise in g.

Section E – The role of population growth

In the previous sections, we modelled the lifetime utility of the whole infinitely-lived household the size of which was growing at the rate of n. This family was concerned also about its future members. Hence, an increase in the growth rate of population n immediately increased saving of this family, leaving the steady-state level of the capital stock per effective worker unaffected (see A7_72D). The same conclusion held for the steady state real interest rate (see A7_48). The economic reason is that the family tries to guarantee the optimum

[75] Recall (see equation no. 45 in the main text) that the steady state nominal rate of interest is $i^* = \rho - n - (1-\theta)g$. Hence its level is unaffected by changes in g, if the utility function is logarithmic (θ=1).

growth rate of consumption for all its members (both present and future) unaltered even after an increase in n. Hence, a sudden increase in the rate of expansion of the family leads to an immediate increase in saving (reduction in present consumption) that will perfectly offset this increase in n (see Figures no. 23_A7 and 24_A7). As a result, the optimum growth rate of consumption of each member (see A7_42) is unaffected after the shock to n.

However, we could have chosen a different modelling strategy (see Blanchard, Fisher: ???:). Let us consider an economy of infinitely-lived individuals, each maximizing his or her life-time utility:

$$U_s = \int_s^{\infty} e^{-\rho\,(\mathrm{t\text{-}s})} \frac{C(t)^{1-\theta}}{1-\theta} dt \qquad \text{(A7\_1 E)}$$

The growth rate of population in this economy is n. Thus, instead of maximizing from a specific time 0, we consider a representative agent standing at time s. Each agent is concerned only about his or her well-being, not about the others. Hence, he or she does not take into account the growth rate of population as in (A7_12).

The structure of the model and all important equations are (almost) the same as before. Yet, the idea behind the flow budget constraint (A7_25) or (A7_31) must be slightly refined. At the aggregate level the private debt is zero, thus the aggregate holdings of assets $B^{agr}(t)$ must be equal to the aggregate stock of capital K(t). Moreover, aggregate assets might increase only due to aggregate savings in the society:

$$\dot{B}^{agr}(t) = W(t)L(t) - C(t)L(t) + r(t)B^{agr}(t) \qquad \text{(A7\_2 E)}$$

W(t)L(t) represents total amount of wages in the economy at time t, C(t)L(t) total consumption at time t. The aggregate interest at time t is reflected by $r(t)B^{agr}(t)$.

The law of motion of individual assets might be easily determined from (A7_2 E), once we realize that the level of individual assets B(t) is equal to $B^{agr}(t)/L(t)$. Hence, the instantaneous change in B(t) is as follows:

$$\dot{B}(t) = \frac{\dot{B}^{agr}(t)L(t) - B^{agr}(t)\dot{L}(t)}{L^2(t)} = \frac{\dot{B}^{agr}(t)}{L(t)} - \frac{B^{agr}(t)}{L(t)}\frac{\dot{L}(t)}{L(t)} = \frac{\dot{B}^{agr}(t)}{L(t)} - n\frac{B^{agr}(t)}{L(t)} \qquad \text{(A7\_3 E)}$$

Inserting (A7_2 E) into (A7_3 E), we get:

$$\dot{B}(t) = \left[\frac{W(t)L(t) - C(t)L(t) + r(t)B^{agr}(t)}{L(t)} - n\frac{B^{agr}(t)}{L(t)}\right] \qquad \text{(A7\_4 E)}$$

Since $B^{agr}(t) = B(t)L(t)$, equation (A7_4 E) yields:

$$\dot{B}(t) = W(t)L(t) - C(t)L(t) + r(t)B(t) - nB(t) \qquad \text{(A7\_5 E)}$$

Notice that this equation is exactly the same as equation (A7_31).

Furthermore, the objective of our representative individual is to maximize (A7_1 E) with respect to the flow budget constraint (A7_5 E). Using similar methods as before, we will arrive at the Euler equation:[76]

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho-n}{\theta} \qquad \text{(A7\_6 E)}$$

As can be seen, the optimum growth rate of consumption per person is negatively affected by the population growth. The economic reason is that our "selfish" individual does not (sufficiently) increase saving immediately after the increase in n. This is the key difference compared with the modelling of the entire family that was perfectly altruistic as regards its future members. As was said before, in case of the family an increase in n did not affect optimum consumption growth (apart from time 0), because the entire shock was absorbed by higher saving at time 0.

With respect to the consumption per effective worker c(t)=C(t)/A(t), (A7_6 E) might be used to show that:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\dot{C}(t)}{C(t)} - \frac{\dot{A}(t)}{A(t)} = \frac{r(t)-\rho-n}{\theta} - g = \frac{r(t)-\rho-n-\theta g}{\theta} \qquad \text{(A7\_7 E)}$$

Thus, the steady state real (natural) interest rate is positively affected by the population growth:

$$r^* = \rho + n + \theta g \qquad \text{(A7\_8 E)}$$[77]

Increase in n is not (fully) reflected by higher saving of existing individuals. As a result, the given capital stock is then split among more individuals. This leads to lower capital per effective worker k and to higher interest rate r (see Figure no. 25_A7 and 26_A7).[78]

This conclusion might be easily proved by deriving the steady state level of capital per effective worker. Using the same procedure as before:

[76] Notice that the only difference is the absence of L(t)/H in the life-time utility function (A7_1 E). Thus, the "exp" term in (A7_35) is without n. As a result, step (A7_41) for example, takes the form:

$$\left[r(t)-n\right] = -\left[(-\rho)+(-\theta)\frac{\dot{C}(t)}{C(t)}\right]$$

[77] Bohm Bawerk (???:???) claimed that the interest rate is positively affected by the undervaluation of future wants, productivity of capital and the population growth. The first phenomenon is reflected by ρ in our model, the second by g and the third by n.

[78] The impact on the optimum initial consumption critically depends on the shape of the saddle path, which in turn is affected by parameter θ. The precise analysis of this path would require similar approximation around the steady state as was performed before. However, the idea is similar to a change in A described in the previous section. Thus, a simple graphical representation seems to be adequate for our purposes. In other words, there is no need to derive the solution of the entire system of differential equation for this minor modification of our model. Since the steady state consumption in this case decreases, higher preference for consumption smoothing (higher θ) is consistent with a drop in present optimum consumption and thus with the saddle path that is closer to the new capital locus. On the other, lower θ might lead to an increase in present optimum consumption.

$$\alpha k^{\alpha-1} - \delta = \rho + n + \theta.g \qquad \text{(A7\_9 E)}$$

We get:

$$k^* = \left(\frac{\alpha}{\rho + \delta + n + \theta.g}\right)^{\frac{1}{1-\alpha}} \qquad \text{(A7\_10 E)}$$

Hence, steady state level of capital is negatively related to n. The optimum saving rate at the steady state is also affected by n:

$$s^* = (n + g + \delta)\frac{\left(\frac{\alpha}{\rho + \delta + n + \theta.g}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{\rho + \delta + n + \theta.g}\right)^{\frac{\alpha}{1-\alpha}}} \qquad \text{(A7\_11 E)}$$

$$s^* = (n + g + \delta)\frac{\alpha}{\rho + \delta + n + \theta.g} \qquad \text{(A7\_12 E)}$$

The specific impact of n on s^* is then given by:

$$\frac{\partial s^*}{\partial n} = \frac{\alpha(\rho + \delta + n + \theta.g) - \alpha(n + g + \delta)}{(\rho + n + \delta + \theta.g)^2} \qquad \text{(A7\_13 E)}$$

$$\frac{\partial s^*}{\partial n} = \frac{\alpha(\rho + \delta + n + \theta.g - n - g - \delta)}{(\rho + n + \delta + \theta.g)^2} \qquad \text{(A7\_14 E)}$$

$$\frac{\partial s^*}{\partial n} = \frac{\alpha[\rho + (\theta - 1)g]}{(\rho + n + \delta + \theta.g)^2} \qquad \text{(A7\_15 E)}$$

The impact of n on saving is thus much lower than in our previous model (see A7_79D):

$$\frac{\partial s^*}{\partial n} = \frac{\alpha}{\rho + \delta + \theta.g} \qquad \text{(A7\_16 E)}$$

In other words, increase in n might increase the optimum saving in the new steady state, but not enough to fully compensate for this increase, since k^* is negatively related to n (and r^* is positively related to n). Thus, higher n will result in lower capital but also in lower consumption per worker (see Figure no. 25_A7 and 26_A7) compared with the analysis of dynastic families (see Figure no. 23_A7 and 24_A7). In other words, not enough saving on the part of selfish individuals after the positive shock to n and a relatively higher present consumption is “penalized” by relatively lower steady state (i.e. future) consumption due to lower steady state capital per (effective) worker.

At the end of this section, let us stress that the RCK economy is dynamically efficient even when we assume selfish infinitely lived individuals. The restriction on convergence of the lifetime utility is as follows:

$$\rho - (1-\theta)g > 0 \qquad \text{(A7\_17 E)}$$

This condition might be derived from (A7_1 E) using the same method as in (A7_13), but neglecting the expansion of the family. Furthermore, (A7_17 E) can be written as:

$$\rho + \theta.g > g \qquad \text{(A7\_18 E)}$$

$$\rho + \theta.g + n > g + n \qquad \text{(A7\_19 E)}$$

Yet, the left hand side of (A7_19 E) is equal to the steady state real interest rate and the right hand side is equal to the growth rate of GDP at the steady state (BGP). Thus, the economy is dynamically efficient and the nominal interest rate is positive even for constant money and velocity, growing economy and gradually decreasing price level.

Section F – Deriving the initial level of consumption from the intertemporal budget constraint

In this section, we will demonstrate that the relative strength of the substitution and the income effect from the increase in the real interest rate depends on parameter θ. Specifically, if θ<1 then the substitution effect dominates and increase in r leads to lower present consumption (or better MPC –marginal propensity to consume). The opposite conclusion holds for θ>1. If θ=1 both effects compensate each other and a hypothetical saving curve is vertical.

First, consider the intertemporal budget constraint (A7_34). Since we focus only on the role of the interest rate, let us assume that population and technology are constant, all households have one member and suppose that both the interest rate and real wage are also time invariant. Thus, the intertemporal budget constraint becomes:

$$\int_0^\infty e^{-r.t} C(t)dt = \frac{K(0)}{H} + \int_0^\infty e^{-r.t} W dt \qquad \text{(A7\_1 F)}$$

The Euler equation that characterises the optimum path of consumption over time (A7_42) might be written as:

$$\frac{d \ln C(t)}{dt} = \frac{r-\rho}{\theta} \qquad \text{(A7\_2 F)}$$

Integrating both sides with respect to time yields:

$$\ln C(t) = \frac{r-\rho}{\theta} t + J \qquad \text{(A7\_3 F)}$$

J is an arbitrary constant of integration. (A7_3 F) gives us:

$$C(t) = \exp(\frac{r-\rho}{\theta}t + J) \qquad \text{(A7\_4 F)}$$

Since at time 0 consumption is C(0), (A7_4 F) implies that exp(J)=C(0). Thus, the solution of (A7_2 F) is simply:

$$C(t) = C(0)\exp(\frac{r-\rho}{\theta}t) \qquad \text{(A7\_5 F)}$$

Substitution of (A7_5 F) to (A7_1 F) gives us:

$$\int_0^\infty e^{-r.t}C(0)e^{\frac{r-\rho}{\theta}t}dt = \frac{K(0)}{H} + \int_0^\infty e^{-r.t}Wdt \qquad \text{(A7\_6 F)}$$

$$C(0)\int_0^\infty e^{-\frac{\rho-(1-\theta)r}{\theta}.t}dt = \frac{K(0)}{H} + W\int_0^\infty e^{-r.t}dt \qquad \text{(A7\_7 F)}$$

The integral on the left-hand side converges, if ρ-(1-θ)r>0:

$$\int_0^\infty e^{-\frac{\rho-(1-\theta)r}{\theta}.t}dt = \left[-\frac{\theta}{\rho-(1-\theta)r}e^{-\frac{\rho-(1-\theta)r}{\theta}.t}\right]_0^\infty = \frac{\theta}{\rho-(1-\theta)r} \qquad \text{(A7\_8 F)}$$

The value of the integral on the right-hand side is:[79]

$$\int_0^\infty e^{-r.t}dt = \left[-\frac{1}{r}e^{-r.t}\right]_0^\infty = \frac{1}{r} \qquad \text{(A7\_9 F)}$$

Hence, (A7_7 F) may be written as follows:

$$C(0) = \frac{W}{r}\frac{\rho-(1-\theta)r}{\theta} + \frac{\rho-(1-\theta)r}{\theta}\frac{K(0)}{H} \qquad \text{(A7\_10 F)}$$

---

[79] Since wages are constant by assumption, real interest rate must be positive to get converging integral. Nonetheless, as will be seen in section G, negative real interest rate can be obtained only with generally diminishing labour income endowment. Thus, assuming positive subjective discount rate and constant wages over time, the general equilibrium requires that the real interest rate is surely positive as well.

The impact of the interest rate on the initial optimum consumption might be expressed as:

$$\frac{\partial C(0)}{\partial r} = \frac{W}{\theta}\frac{-(1-\theta)r-\rho+(1-\theta)r}{r^2} - \frac{(1-\theta)}{\theta}\frac{K(0)}{H} = -\frac{W}{\theta}\frac{\rho}{r^2} - \frac{(1-\theta)}{\theta}\frac{K(0)}{H} \qquad \text{(A7\_11 F)}$$

This expression is quite complicated and can generate a non-linear re-switching relationship between rising interest rate and present consumption (see Figure no. 27_A7 for θ=2). This might be explained by the impact of r on the present value of the flow of wages W/r. On the other hand, C(0) is easy to determine, if ρ = r:

$$C(0) = \frac{W}{r}\frac{r-(1-\theta)r}{\theta} + \frac{r-(1-\theta)r}{\theta}\frac{K(0)}{H} \qquad \text{(A7\_10 Fb)}$$

$$C(0) = \frac{W}{r}\frac{r-r+\theta.r}{\theta} + \frac{r-r+\theta.r}{\theta}\frac{K(0)}{H} \qquad \text{(A7\_10 Fc)}$$

$$C(0) = W + r\frac{K(0)}{H} \qquad \text{(A7\_10 Fd)}$$

$$C(0) = W + \rho\frac{K(0)}{H} \qquad \text{(A7\_10 Fe)}$$

This level of consumption is then chosen every period onwards, since the growth rate of optimum consumption is zero in this case (see A7_2 F and A7_5 F). Furthermore, this particular level is chosen regardless of θ and might be easily indicated in Figure no. 27_A7 as the intersection of all the curves reported. Another interesting aspect is the size of this consumption. It is equal to the sum of the every-period wage and the permanent dividend obtained from the capital assets. Since the entire labour and capital incomes are consumed every period, savings are zero and hence assets are also constant over time being permanently maintained at the initial level of K(0)/H.

Furthermore, expression (A7_10 F) is simplified also for θ=1:

$$C(0) = \frac{\rho}{r}W + \rho\frac{K(0)}{H} \qquad \text{(A7\_10 Ff)}$$

Figure no. 27_A7 clearly indicates that in this case the optimum consumption declines with the increasing real interest rate along a rectangular hyperbola.

Notice that the present consumption is depressed to zero, if r = 6 %, θ = 3 % and ρ = 3 % (see Figure no. 27_A7). However, this combination is ruled out by condition ρ-(1-θ)r>0, which guarantees convergence of the "consumption integral" in (A7_7 F). Thus, optimum present consumption cannot be zero.

Furthermore, if we set real wage to zero, or if we neglect the impact of r on the present value of the flow of wages and denote W/r + K(0)/H as the "Wealth" of this consumer, we get:

$$C(0) = \frac{\rho-(1-\theta)r}{\theta}Wealth \qquad \text{(A7\_12 F)}$$

ρ/θ - $r(1-\theta)/\theta > 0$ can be interpreted as the marginal propensity to consume out of wealth MPC_{Wealth} (Barro 2004:94). As can be seen, present consumption is not depressed to zero even for zero time preference (in sense two, i.e. $\rho=0$) and for positive real interest rate, provided that the elasticity of substitution is low enough ($\theta>1$). This conclusion was derived many times before, but it applies also for the continuous time model. Hence, we proved again that Mises was not right in this respect.

Furthermore, the sensitivity of C(0) with respect to the interest rate is:

$$\frac{\partial C(0)}{\partial r} = -\frac{(1-\theta)}{\theta} Wealth \qquad \text{(A7\_13 F)}$$

The response of present consumption (or better just the MPC_{Wealth}) to the interest rate critically depends on θ (see A7_13 F). If the preference for consumption smoothing is significant ($\theta>1$), an increase in the interest rate will also raise present consumption. In such a case, we can say that the saving curve is downward sloping. On the other hand, low preference for consumption smoothing ($\theta<1$) results in a decrease in present consumption (or better MPC_{wealth}) after the rise in the interest rate. In this case, the substitution effect dominates and the saving curve is upward sloping (keeping wealth constant).

Section G – An economy gradually approaching zero natural rate of interest

In this section, we briefly discuss an economy that gradually reaches zero real natural rate of interest. Consider an economy with Cobb-Douglas production function with $\alpha = 1/3$, subjective discount rate of 5 %, logarithmic utility ($\theta=1$), zero population growth, depreciation rate of 6 % and positive technological progress of 2 %.

According to (A7_48), the steady state natural rate of interest might be zero, if:

$$r^* = 0 \Leftrightarrow \rho = -\theta.g \qquad \text{(A7\_1 G)}$$

Hence, the necessary technological decay to obtain zero steady state real interest is as follows:

$$r^* = 0 \Leftrightarrow g = -\frac{\rho}{\theta} \qquad \text{(A7\_2 G)}$$

For our set of parameters this implies g = - 5 %. Figure no. 28_A7 depicts the evolution of the real rate of interest after the fall in the technological progress from +2 % to – 5 %. Assuming constant money and velocity, Figure no. 28_A7 also demonstrates that the nominal interest rate never falls to zero or even below zero. Notice that according to equation (46) in the main text, our set of parameters (ZPG and logarithmic utility) implies that the steady state nominal interest rate is equal to the subjective discount rate.

Furthermore, Figure no. 29_A7 and 30_A7 illustrate the behaviour of the optimum saving rate and the growth rate in output per worker. As can be seen, the economy is never dynamically inefficient as the growth rate in GDP is always lower than the real interest rate. Furthermore, because the real interest rate is lower than the subjective discount rate, the optimum consumption will be decreasing over time; on the BGP at the rate of g = – 5 % (see A7_42 or A7_43, and A7_58).

An interesting behaviour can be observed with respect to the saving rate. First, it increases sharply immediately after the shock. The economic reason lies in the fact that people are preparing for lower future income endowment associated with the expected technological decay. This saving behaviour leads to the fact that the economic decline in the transition period is not as fast as the rate of technological decay. This might be seen in Figure no. 30_A7 where the growth rate in output is for many years considerably greater than the fall in technological progress.

However, optimum saving rate gradually falls to much lower levels than we observed before the shock due to relatively low θ (see A7_82D). This could be explained by an upward sloping saving curve and shrinking investment curve that results from technological decline.

Moreover, according to (A7_79D), we can even find a set of parameters that lead to zero optimum steady state saving rate:

$$s^* = (n+g+\delta)\frac{\alpha}{\rho+\delta+\theta.g} = 0 \Leftrightarrow n+g+\delta = 0 \qquad \text{(A7\_3 G)}$$

For our set of parameters, this could be achieved with ZPG, g = - 5 % and the depreciation rate of 5 %. If we look at Figure no. 26_A7, this implies that the break-even-investment line (n+g+δ)k is horizontal, as it coincides with the k-axis. Thus, the eventual picture of the economy closely resembles Figure no. 0b_A7. It should be stressed that even though levels of capital and consumption gradually fall to zero in the infinite horizon, since both variables are decreasing at the rate of – 5 %, according to (A7_72D) and (A7_73D), k* and c* are still positive.

Furthermore, even a negative steady state saving rate is possible, if the depreciation rate is low enough (e.g. δ = 4 %). This would mean that it is optimal to consume capital directly. The economic reason behind this peculiar result is as follows. Since the BGP growth rate in capital is g = - 5 %, deprecation of capital lower than 5 % requires its direct consumption. In our one-good model, in which capital and consumption goods are represented by the same commodity, this is certainly possible. This in economic terms means that gross investment is negative. Yet, in the real world, direct consumption of capital is only a minor phenomenon and all important (planned or unplanned) decreases in capital take the form of the excess of depreciation over positive gross investment.[80] Thus, a straightforward restriction that might be immediately imposed on our model is as follows:

$$s^* = (n+g+\delta)\frac{\alpha}{\rho+\delta+\theta.g} > 0 \Leftrightarrow n+g+\delta > 0 \qquad \text{(A7\_4 G)}$$

As we will see below, the denominator of (A7_4 G) must be positive by assumption. Assuming ZPG, condition (A7_4 G) simply requires that the technological decay must not be larger than the depreciation rate. This condition in turn implies that the gross investment is always positive and capital is never directly consumed.

Even if we allow for direct consumption of capital, the rate of technological decay in this model has a limit due to the non-negativity constraint imposed on k(t) (see A7_10g). From the steady state formula of k* (see A7_72D), we can see that:

$$k^* = \left(\frac{\alpha}{\rho+\delta+\theta.g}\right)^{\frac{1}{1-\alpha}} \geq 0 \Leftrightarrow \rho+\delta+\theta.g \geq 0 \qquad \text{(A7\_5 G)}$$

Moreover, since 0<α<1 due to conditions (A7_10b) - (A7_10f), the denominator must be positive for (A7_72D) to be permissible. Because the steady state real interest rate is r*=ρ+θg, condition (A7_5 G) requires that:

[80] Consumption of capital is thoroughly discussed in Hayek (192???: Consumption of capital) and in Hayek (PTC), Hayek (Maintenance of Capital), …

$$r^* + \delta \geq 0 \qquad \text{(A7\_6 G)}$$

(A7_6 G) simply states that the marginal product of capital at the steady state must not be negative (recall that MPK* = r* + δ). However, assumptions imposed on the production function (see A7_3 and A7_5 for the extensive form and A7_10c and A7_10f for the intensive form) require that the marginal product of capital is always positive. Hence, (A7_5 G) and (A7_6 G) must hold with strict inequality.

We may conclude that the negative saving rate (and negative real natural rate of interest as well) is technically possible in this model owing to negative technological progress. Yet, this negativity has its limits due to the non-negativity constraint imposed on k(t). Condition (A7_5 G) and the positivity of MPK imply that the maximum rate of technological decay is given by:

$$g > \frac{-\rho - \delta}{\theta} \qquad \text{(A7\_7 G)}$$

For our set of parameters, $\rho = 5$ %, $\theta=1$ and $\delta = 6$ %, the limiting g is – 11 %.[81] Furthermore, if we disallow direct consumption of capital, the maximum technological decay is (see A7_4 G):

$$s^* > 0 \Leftrightarrow g > -(n + \delta) \quad \text{(A7\_8 G)}$$

Condition (A7_8 G) then requires that limiting g is – 6 %.

Nonetheless, in our benchmark example $\delta = 6$ % and g = - 5 %. Thus, gross investment and the optimum saving rate on the BGP must be positive to reach a year-to-year decline in capital of only g = - 5 %.

In the main text we focused our major investigation on the zero natural real interest rate. Assuming positive subjective discount rate, this might be achieved only with a rapid technological decay. Yet, in the previous paragraphs we found out that other interesting phenomena might emerge as well, if the economy is gradually contracting. We also determined necessary limits that must be imposed on this technological decline.

Section H – Behaviour of the natural rate of interest if the technological progress is stochastic

In this section, we assume that the technological level A follows a simple stochastic AR(1) or AR(2) process:

$$\ln A_t = \beta_1 \ln A_{t-1} + \beta_2 \ln A_{t-2} + \varepsilon_t \qquad \text{(A7\_1 H)}$$

β_i´s are autoregressive coefficients that measure the degree of memory of this process. If $\beta_2 = 0$, the process is AR(1) approaching random walk for $\beta_1 = 1$. ε_t is the random disturbance having the properties of the white noise.

Figure no. 31_A7 shows the behaviour of the real natural rate of interest it the level of technologies A follows AR(1) process. Since $\beta_1 < 1$, this process is stationary. Nominal

[81] In this case, the break-even investment line in the Solow model will be downward sloping. Then, the resulting negativity of the optimum saving rate is gigantic.

interest reflects a theoretical value, if the growth rate in output is fully reflected (with the opposite sign) in the inflation rate and this in turn in the nominal interest rate. This is the reason for its higher volatility compared with the real rate.

Such fluctuations in the nominal interest rate would be required to keep money neutral with respect to the real economy. In such a case, ex post and ex ante real interest rate will be equal. *However, since shocks are unpredictable, nominal interest rate cannot reflect them at the moment they occur. On the other hand, we depict the behaviour of the economy on a year-to-year basis, but the shock will cause deviation between ex post and ex ante just at that instantaneous moment. In the subsequent periods within the year, the nominal interest rate can freely adjust. Nevertheless, the resulting fluctuations of the nominal interest rate seem to be rather high. One may wonder whether the real world conditions could deliver this necessary volatility. First, contracts between debtors and creditors are agreed in nominal terms that might be fixed for a considerable period of time. Second, commercial banks usually do not change their interest rates so often and to such large extent. And finally, modern central banks follow some rule using nominal interest rate as the main policy tool.*

The first reason stressing the rigidity of the nominal interest rate implies that constant money cannot deliver consistency between ex post and ex ante real interest rate and may cause disturbances between actual real interest rate and the real natural rate. Thus, it seems at first glance that better policy would be to aim at price stability by manipulating either the money supply or the nominal interest rate, which may deliver greater consistency between the theoretically optimal real interest rate and the actual real interest rate. This problem will be discussed in a greater detail in Chapter 4 and it was analysed also in Chapter 2.

However, first we must realise the frequency and magnitude of productivity shocks. Suppose that they occur once a year at the magnitude shown in Figure 39_A7. Their maximum impact on the growth rate in output is at one single moment of the shock. Yet, Figure no. 7_A7 clearly shows that the subsequent impact on the real growth rate is much lower. Hence, the influence on the increase or decrease in prices and consequently on the nominal interest rate, assuming constant money and velocity, is far smaller than suggested in Figure 34_A7. Hence, there are self-stabilizing forces in the economy that will significantly dampen the effects of the technological shock in subsequent periods. As a result, the resulting optimum volatility of the nominal interest rate is much lower. Yet, this implication assumes again a relative flexibility in prices and the nominal interest rate so as to reflect the fact that within the same year the period-to-period conditions that follow the shock are very similar.

On the other hand, we can also ask whether the monetary policy that is aimed at the price stability could deliver optimal doses of money that perfectly adapt to fluctuations in output. Since only with such a policy, fluctuations in the nominal interest rate would accurately accord with the volatility of the real natural rate. Furthermore, monetary policy using the nominal interest rate as the major tool should adapt not only to the evolution of the real natural rate of interest but also to all changes in the inflation rate that are provoked by changes in the growth rate in potential output. One may ask whether any monetary policy conducted by erring human beings is capable to perform such a large degree of sophistication.

Hence, constant money (and constant velocity) may represent the ideal policy that would deliver the greatest possible consistence between actual real interest rate and the natural real *interest rate, if the nominal rate of interest is allowed to reflect changes in fundamental variables in the economy.* [82]

Let us focus again on the behaviour of real variables. Figures no. 32_A7 - 34_A7 demonstrate a cyclical behaviour of the real interest rate, saving rate and investment in this simple RBC model.[83] As can be seen, real interest rate is pro-cyclical in this model. At the same time, it is always higher than the growth rate in output, which implies dynamic efficiency. Saving rate is pro-cyclical too due to relatively low θ. Finally, Figure no. 32_A7 clearly shows that investment is much more volatile than GDP in this RBC model.

It can be add that even though the RBC fluctuations are rather mild, the volatility of desired real investment is rather high. Thus, the real interest rate must be flexible enough to reflect these fluctuations and to convey information about these fluctuations.

Figures no. 35_A7 - 37_A7 reveal that the key variables fluctuate less with lower intertemporal substitution of consumption (higher θ). The key economic reason is that with higher preference for consumption smoothing, the response of consumption is not so significant. In other words, saving does not react much to changes in the interest rate. **Figure no. 35_A7B** displays the correlation coefficients between the saving rate and the real interest rate for various θ.

Obviously, the only exception is the real natural rate of interest itself. Since higher θ is associated with less elastic saving curve, real interest fluctuate more in this case. It can be shown that fluctuations in GDP are very similar for various θ. This implies that lower intertemporal substitution is associated with a-cyclical behaviour of the saving rate.

We can even allow the subjective discount rate to follow a stochastic process. Consider the following MA(1) process:

$\rho_t = 0.05 + \mu_t + \gamma.\mu_{t-1}$ (A7_2 H)

Parameter γ measures the influence of the previous shock on the present shock. Figure no. 38_A7 demonstrates that the volatility of the natural rate of interest is lower than the volatility of the subjective discount rate.

If both the technological level and the subjective discount rate are stochastic, the picture of such an economy might be represented by Figure no. 39_A7. This figure could reflect a typical behaviour of the natural rate of interest that is affected both by changes in the subjective discount rate and by changes in productivity.[84] We obviously assume that shocks to technology and to the time preference are uncorrelated.

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[82] We are obviously neglecting the crucial time dimension in this process. At one moment the credit contract is negotiated, at some future moment, it is settled. The eventual level of the natural rate of interest could be totally different. Obviously, the entire optimization problem should be redefined if we allow for uncertainty in the model. Euler equations will be modified due to the presence of the stochastic element. In our approach, we only analyze optimum response of present consumption to an unpredictable change in the real interest rate. This will in turn affect future accumulation of capital, marginal product of capital and hence the natural rate of interest.

[83] Obviously, one of the main building blocks of the RBC theory, intertemporal substitution of labour, is missing in our model. Yet, we are mainly interested in the optimum intertemporal consumption behaviour and the resulting natural rate of interest.

[84] The subjective discount rate, reflecting the desire to achieve the given want sooner rather than later, affects rather the next period interest rate in our model.

Appendix 7 Figures

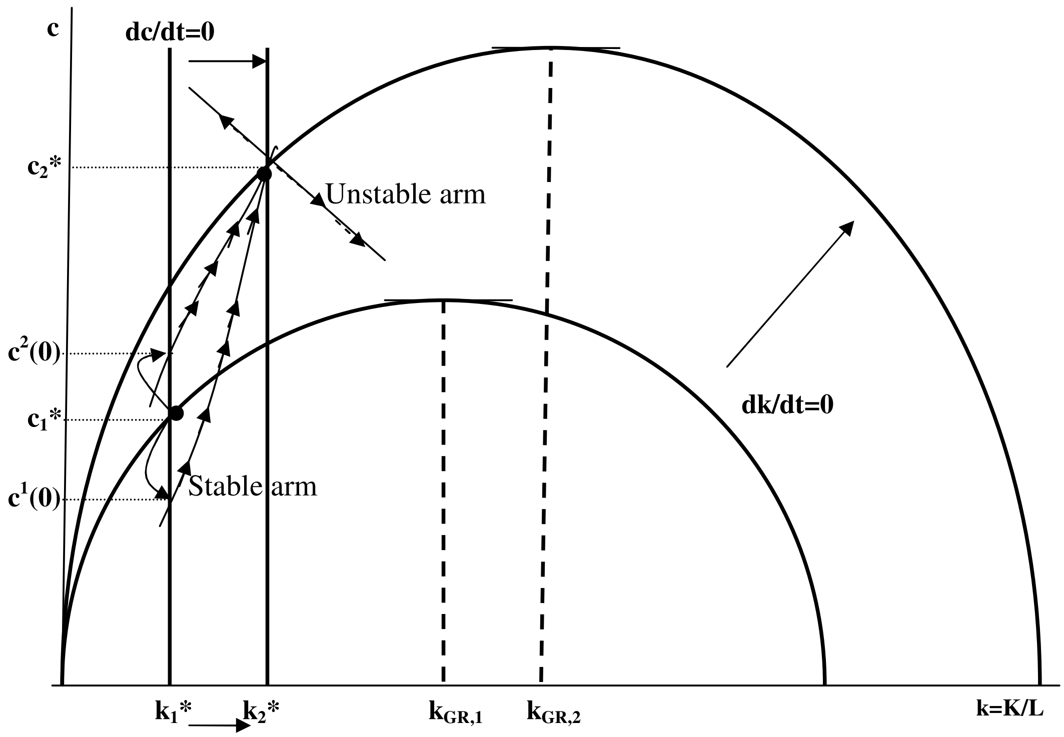

Figure no. 1_A7 Increase in A in the RCK model.
Note: $c^1(0)$ is the initial optimal consumption for low θ, $c^2(0)$ is the initial optimal consumption for high θ. k_1^* performs the role of k(0), if the relevant steady state is k_2^*

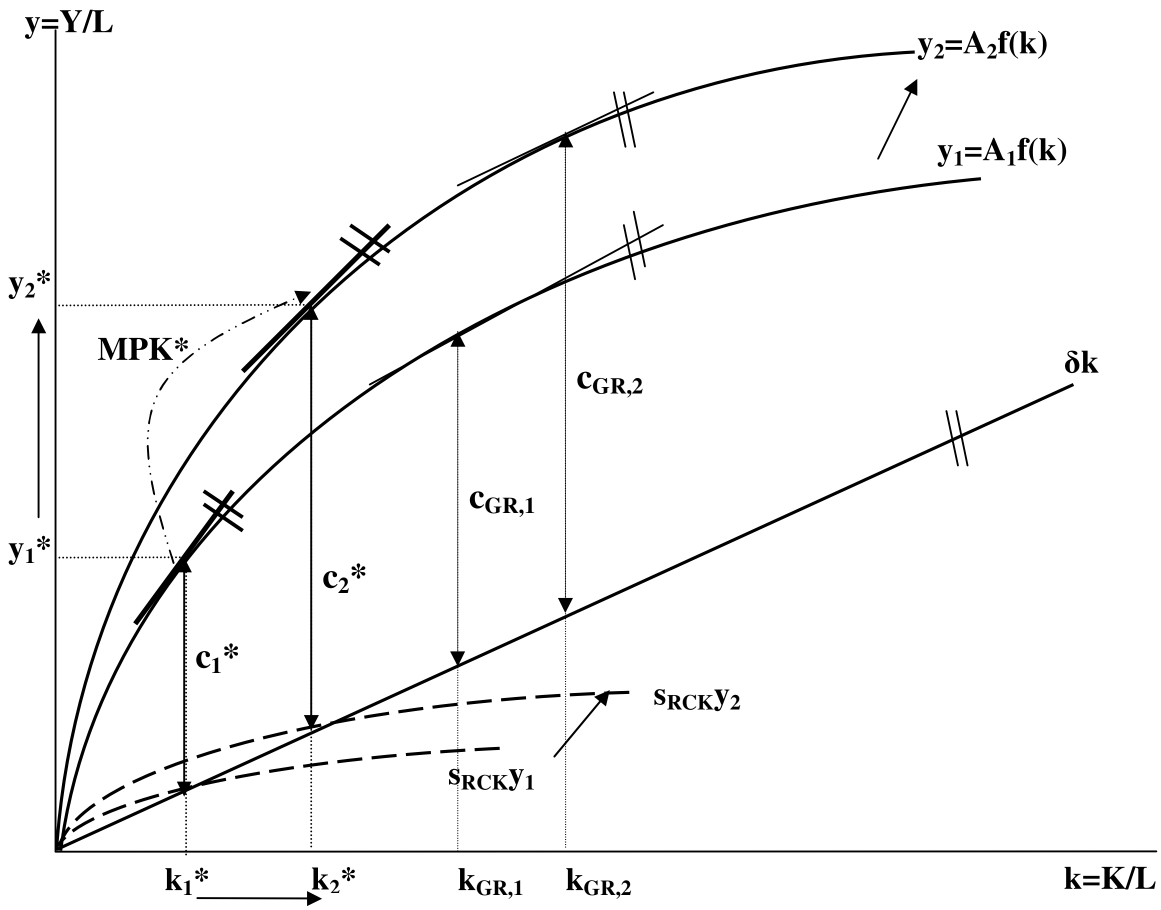

Figure no. 2_A7 Increase in A in the RCK model represented in the Solow model.
Note: c_1^* is definitely lower than c_2^*. However, optimum saving rate is the same in both steady states(s_{RCK}). The same holds for the steady state natural rate of interest $r^*=MPK^*-\delta$, which is determined solely by the subjective discount rate ρ. Thus, the slope of the production function at the steady state, which represents MPK, is equal in both steady states. Golden rule might be found at the point where the slope of the production function (MPK) is equal to the slope of the depreciation curve (δ).*

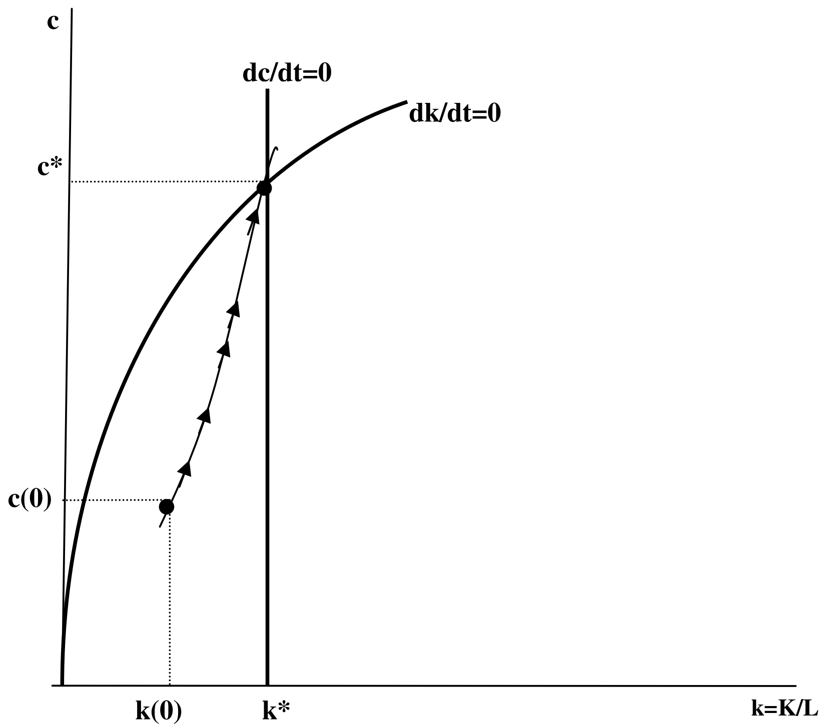

Figure no. 0a_A7 RCK model for n=g=δ=0

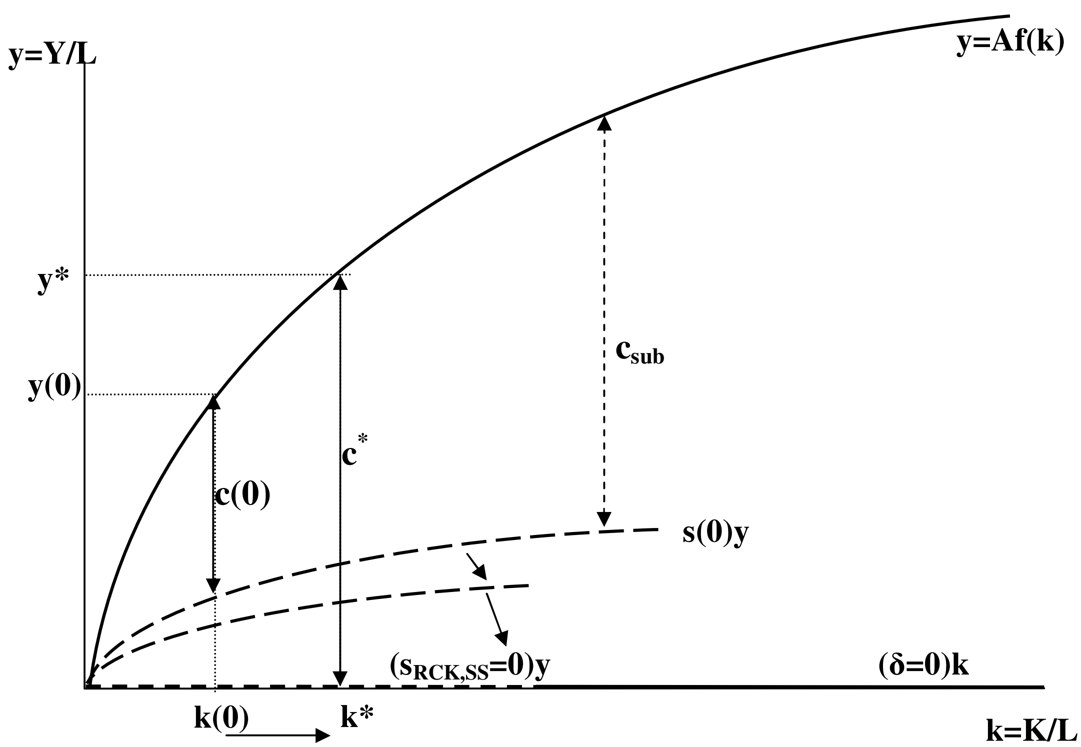

Figure no. 0b_A7 Solow model representation of the RCK model for n=g=δ=0
Note: Optimum saving rate gradually falls from s(0) to $s_{RCK,SS}=0$.

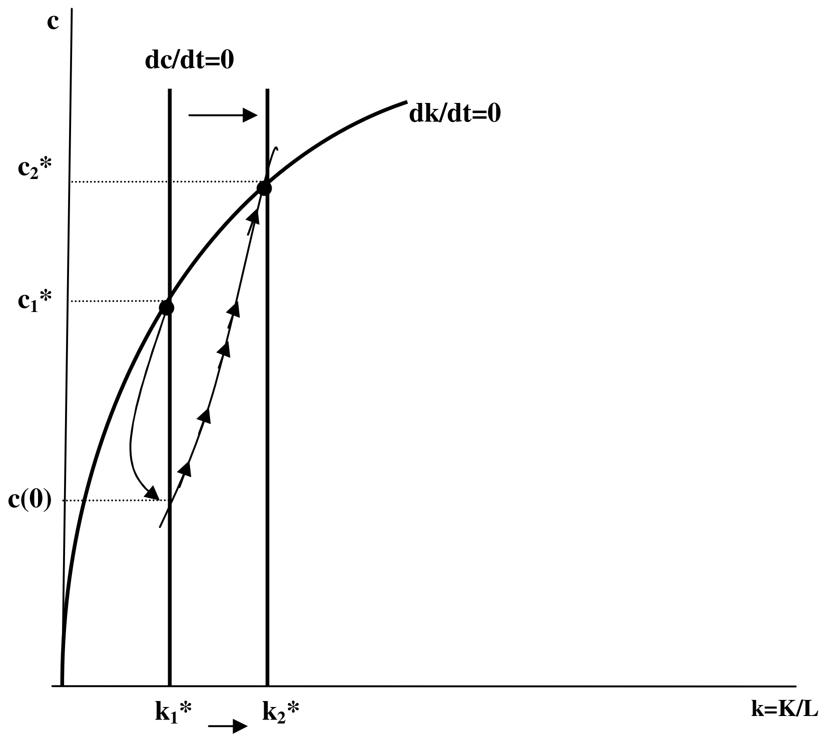

Figure no. 0c_A7 Decrease in ρ in the RCK model for n=g=δ=0.

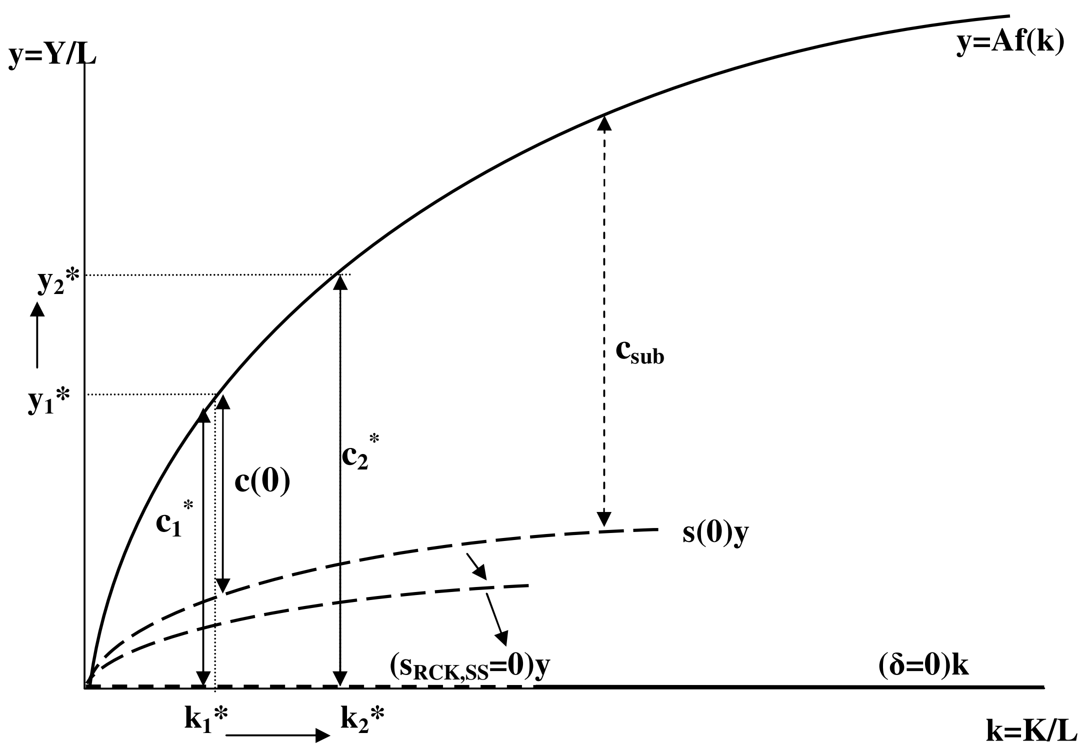

Figure no. 0d_A7 Solow model representation of the decrease in ρ in RCK model for n=g=δ=0
Note: Optimum saving rate after the decrease in ρ *gradually falls back to $s_{RCK,SS}$=0.*

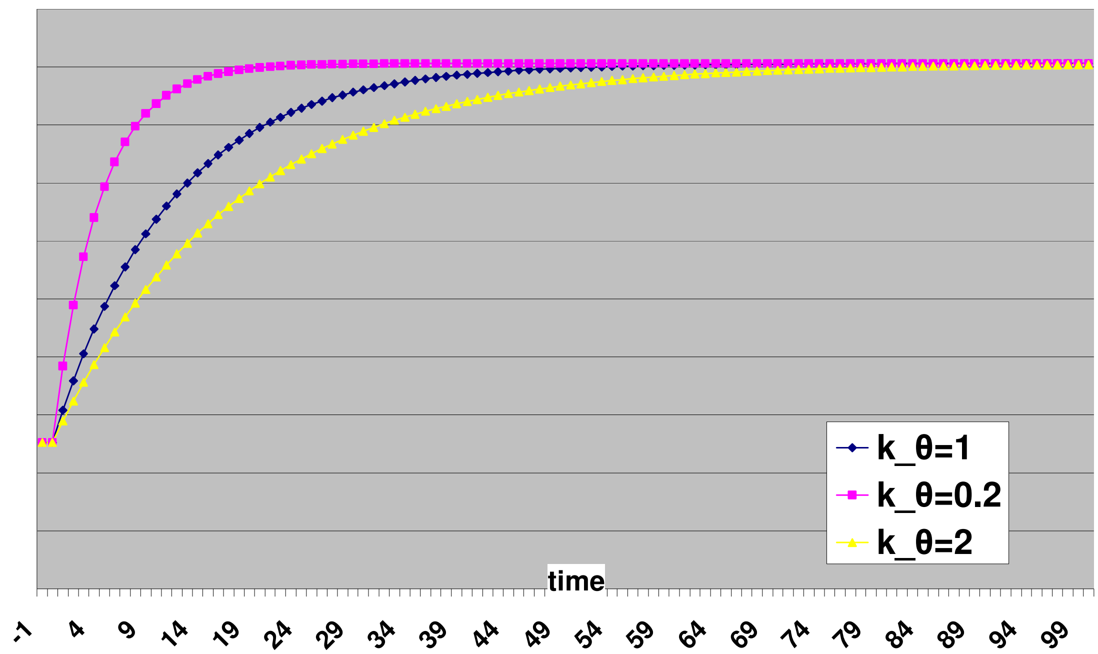

Figure no. 3_A7 Evolution of capital per worker after the increase in A in the RCK model and the role of θ.

Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

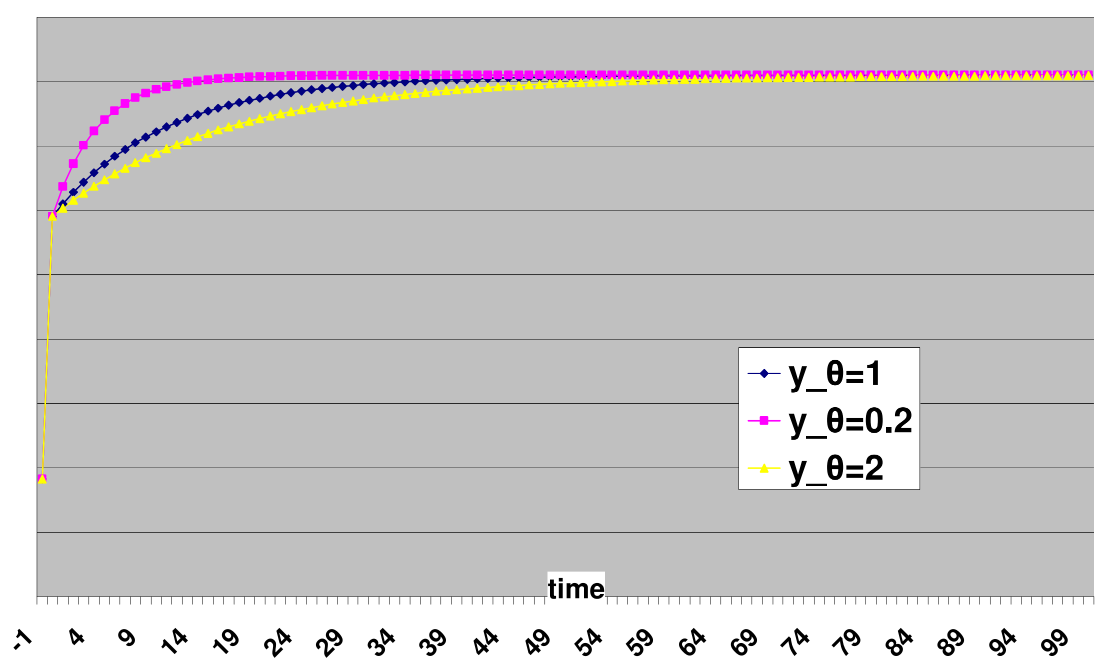

Figure no. 4_A7 Evolution of output per worker after the increase in A in the RCK model and the role of θ.

Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

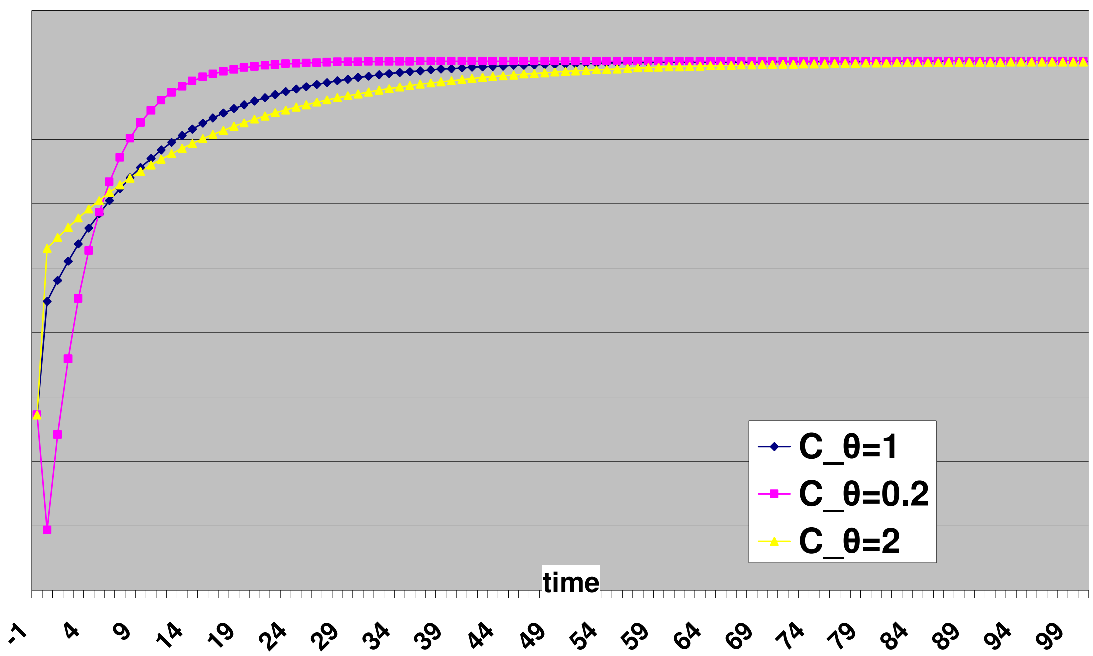

Figure no. 5_A7 Evolution of consumption per worker after the increase in A in the RCK model and the role of θ.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

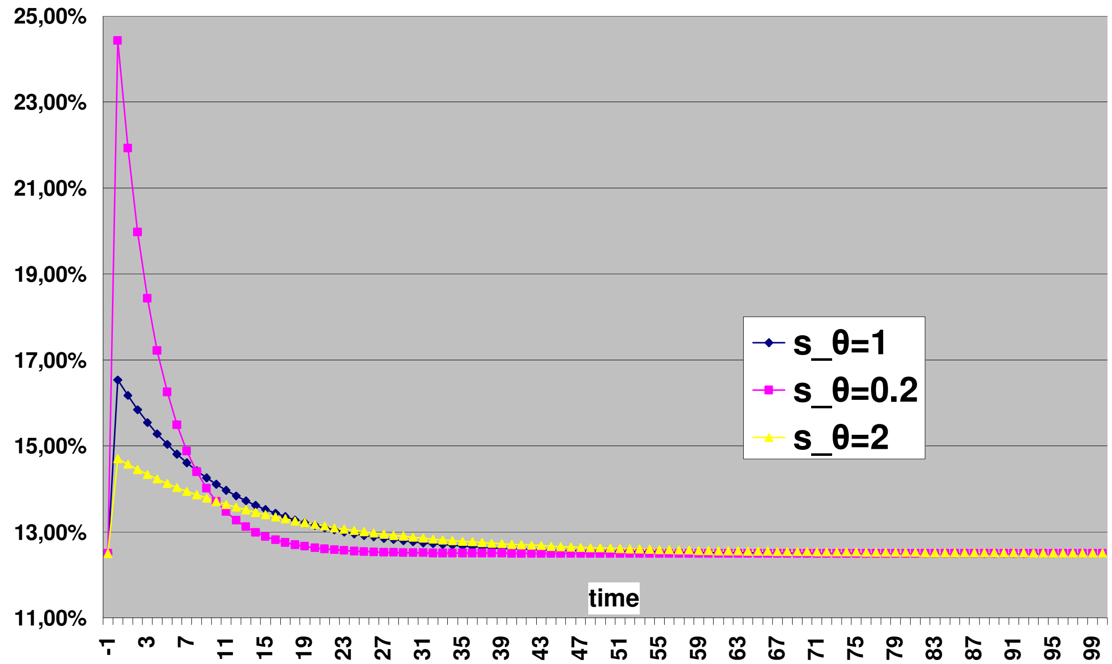

Figure no. 6_A7 Evolution of the saving rate after the increase in A in the RCK model and the role of θ.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

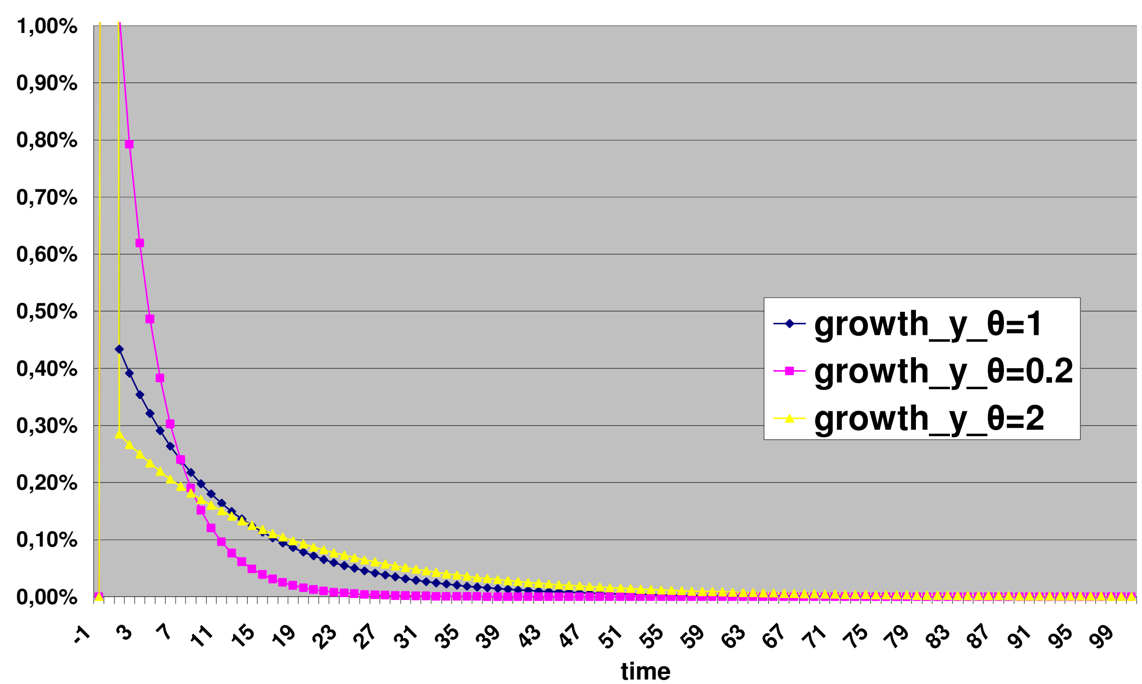

Figure no. 7_A7 The growth rate in output per worker after the increase in A in the RCK model and the role of θ.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

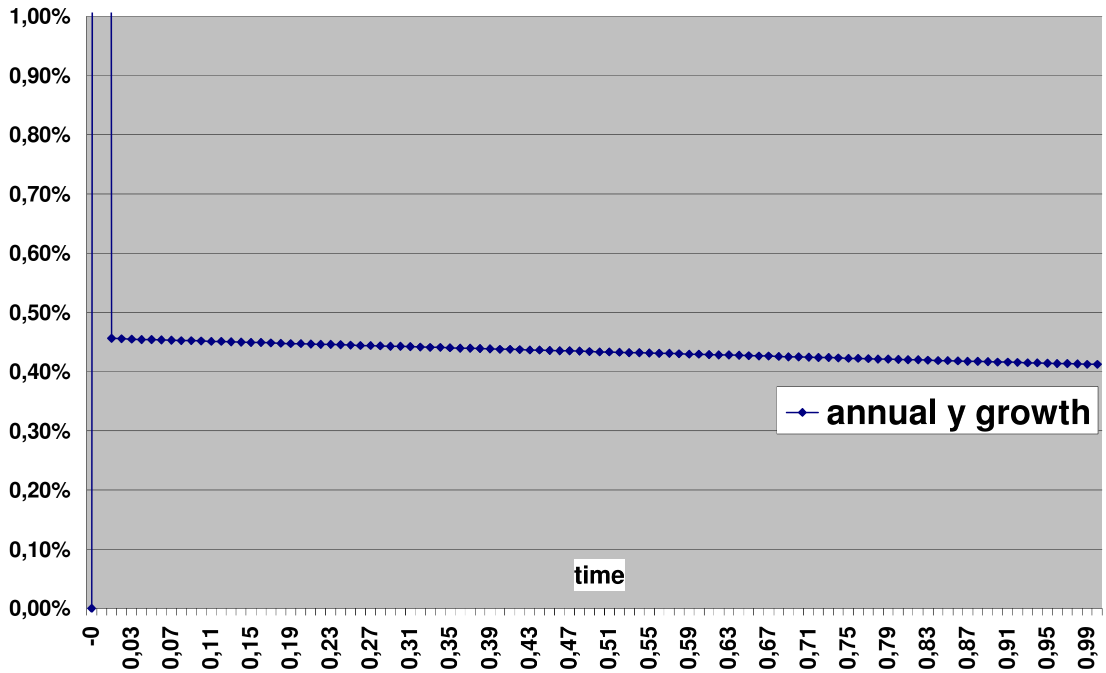

Figure no. 7_A7b The growth rate in output per worker within one year after the increase in the level of technologies A.

Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %, θ =1.

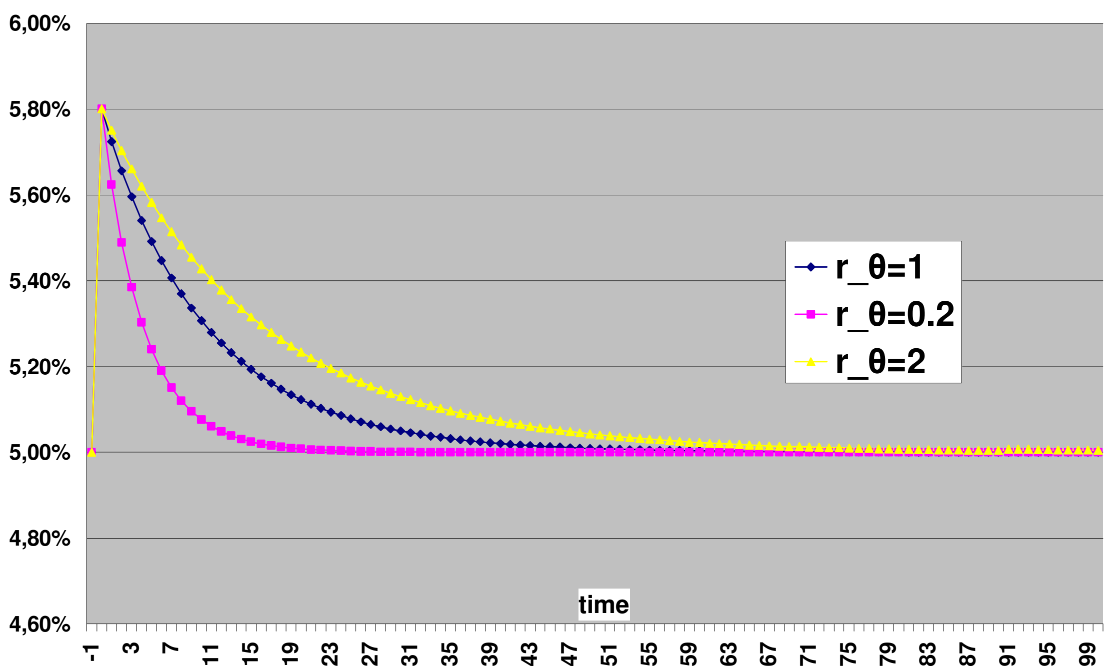

Figure no. 8_A7 The real interest rate after the increase in A in the RCK model and the role of θ.

Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %

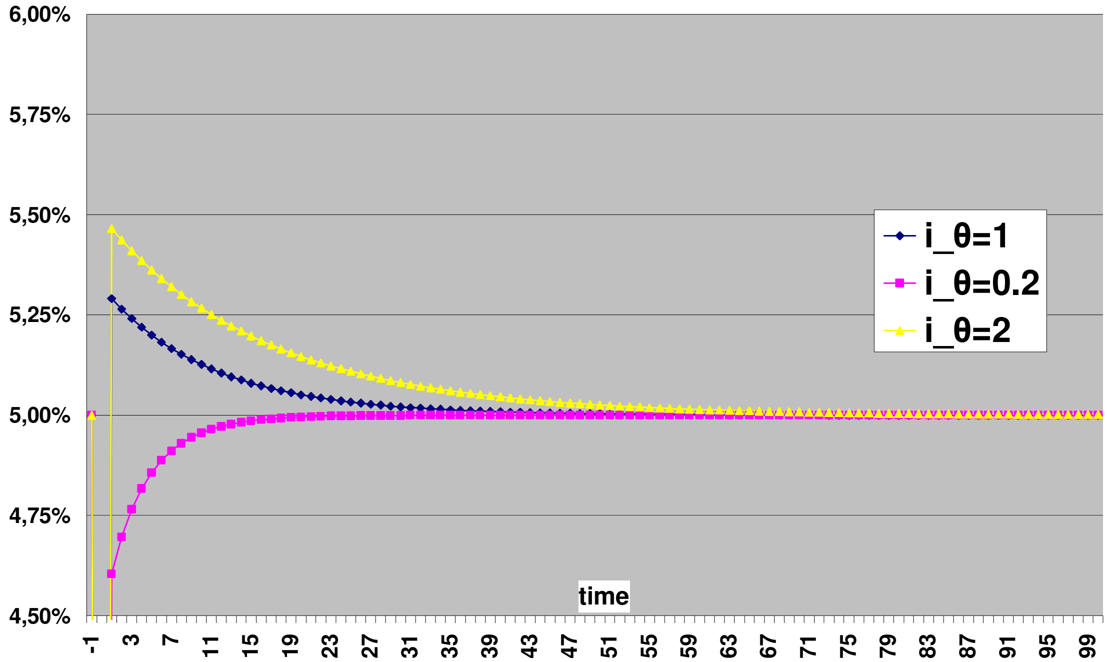

Figure no. 9_A7 The nominal interest rate after the increase in A in the RCK model and the role of θ.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, ρ=5 %. Money and velocity are constant.

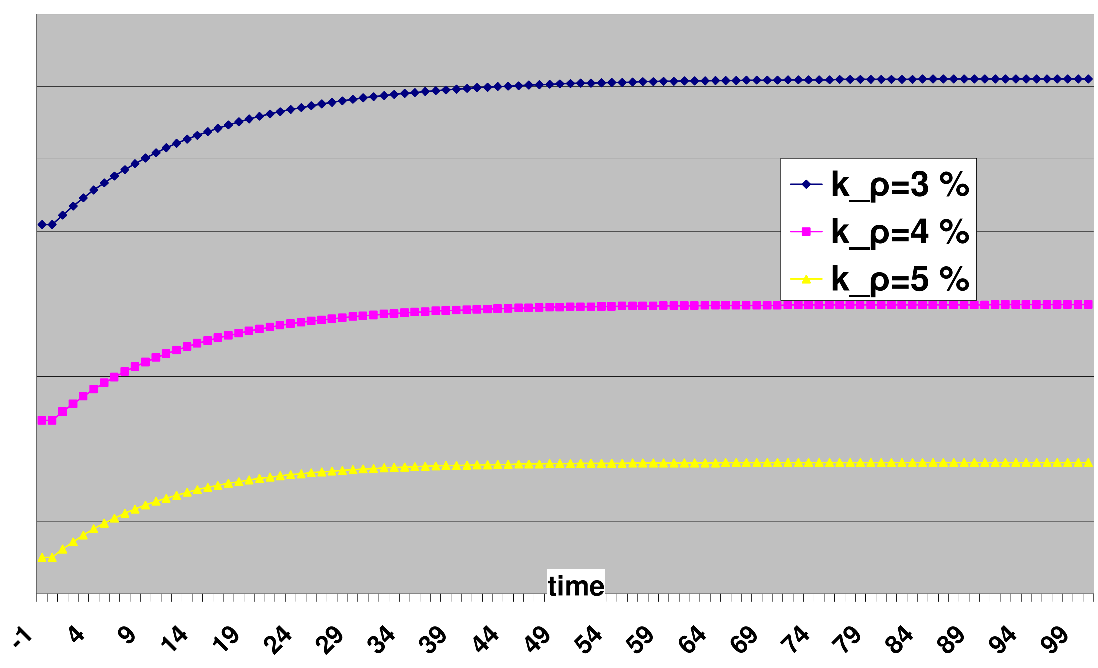

Figure no. 10_A7 Evolution of capital per worker after the increase in A in the RCK model and the role of ρ.

Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, θ =1.

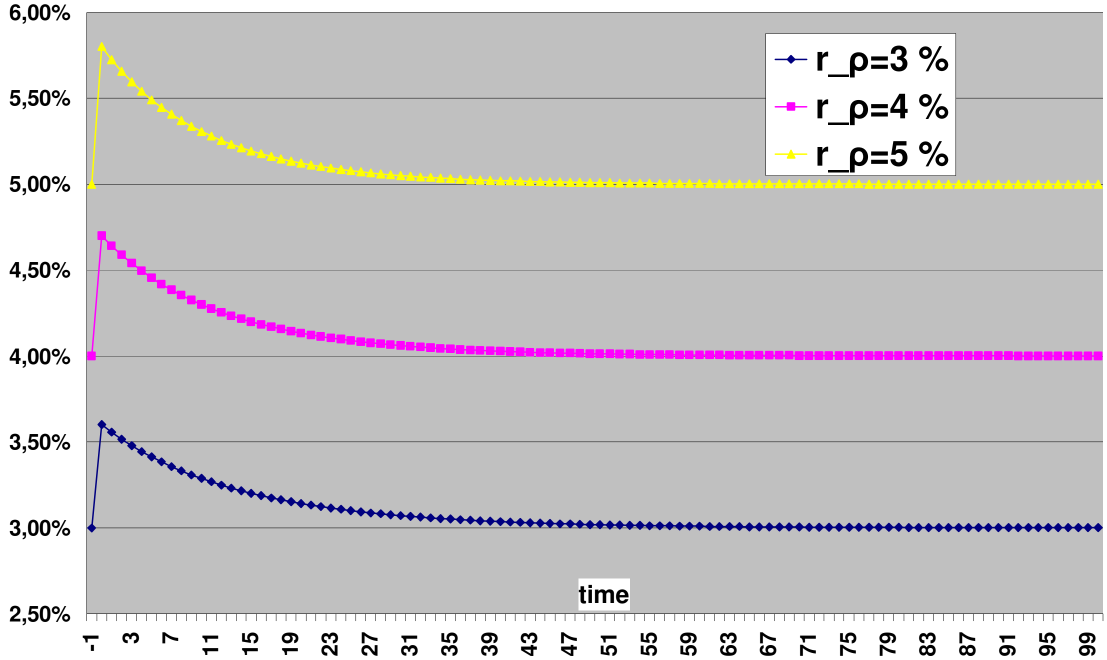

Figure no. 11_A7 The real interest rate after the increase in A in the RCK model and the role of ρ.
Note: Parameter A permanently rises from 1 to 1.1 (i.e. by 10 %) at time 0. The set of exogenous parameters is as follows: δ= 3 %, α=1/3, θ =1.

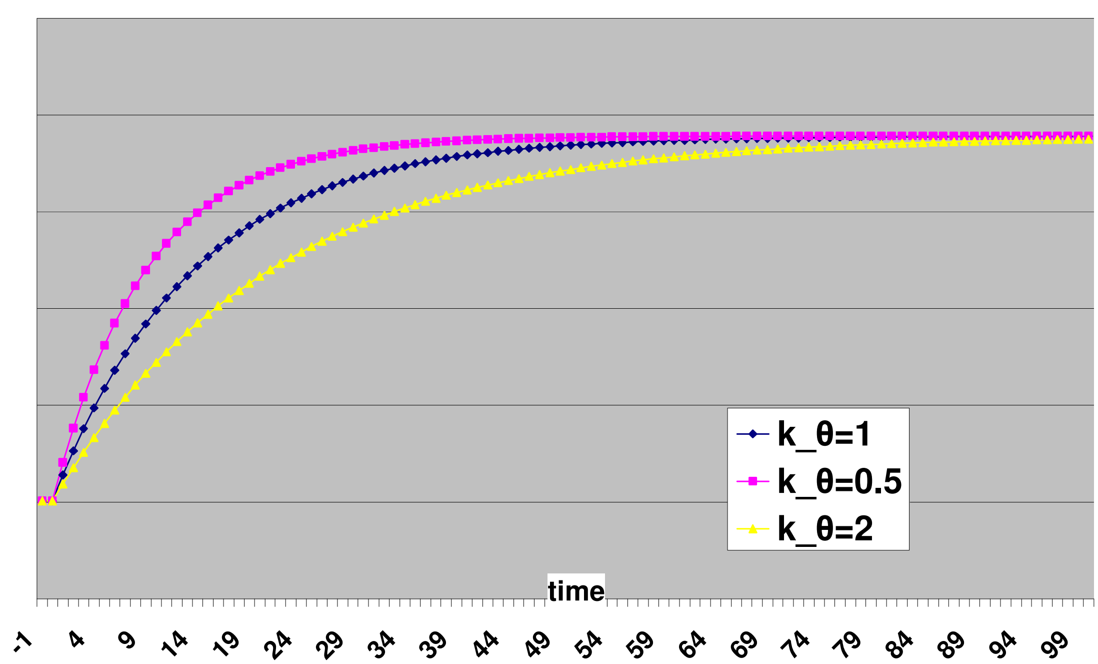

Figure no. 12_A7 Evolution of capital per worker after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3

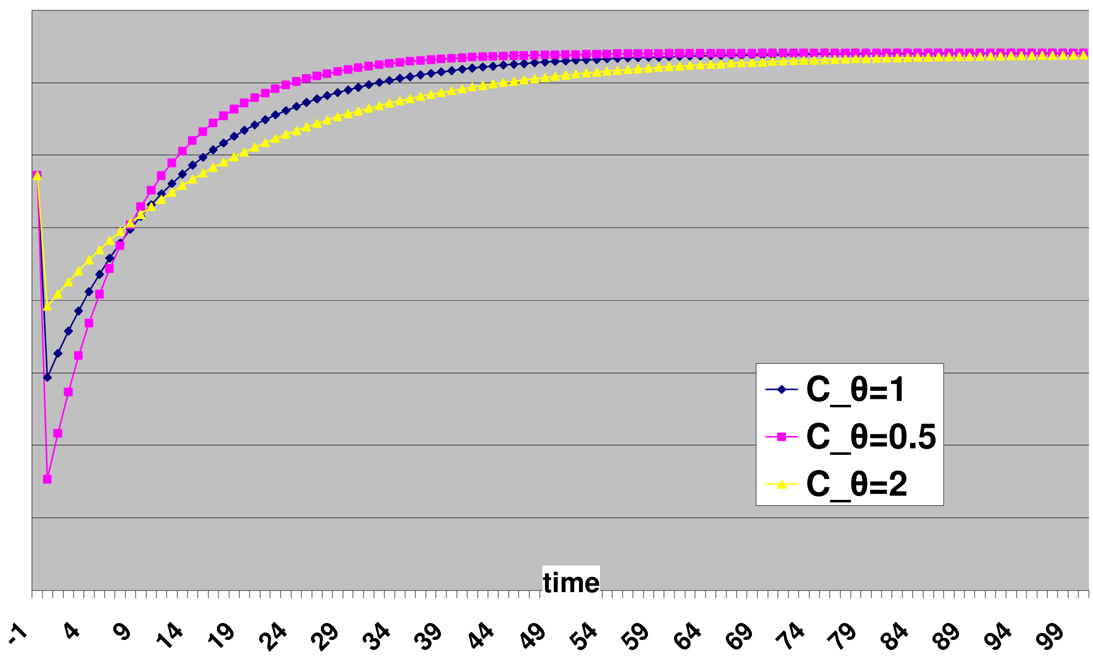

Figure no. 13_A7 Evolution of consumption per worker after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3

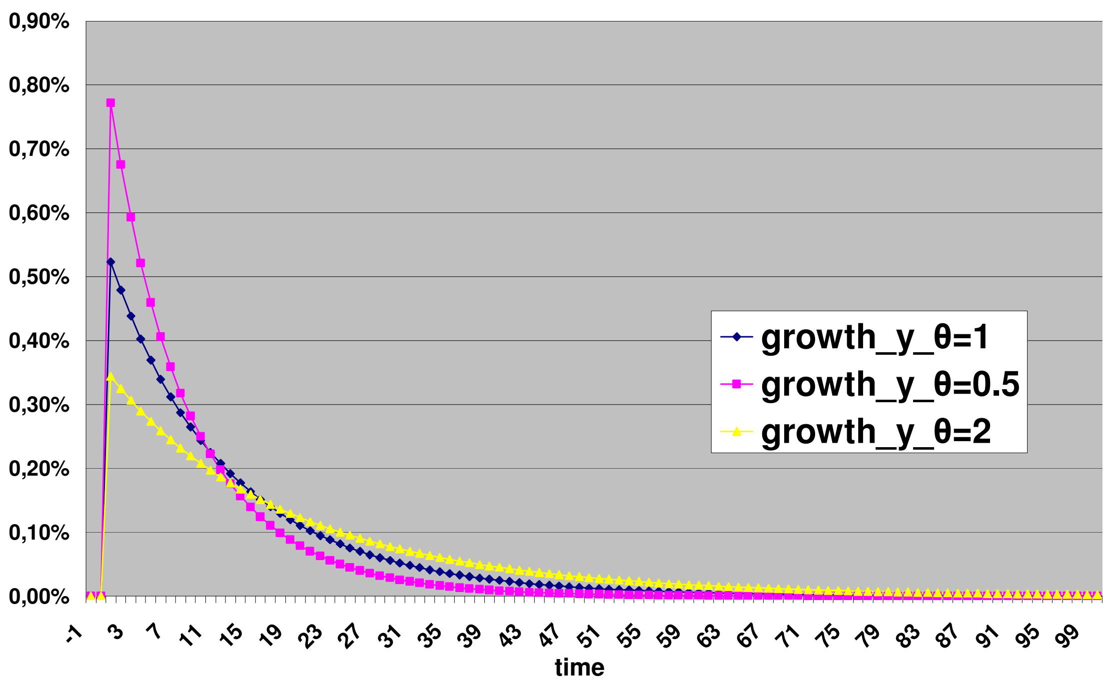

Figure no. 14_A7 The growth rate in output per worker after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3

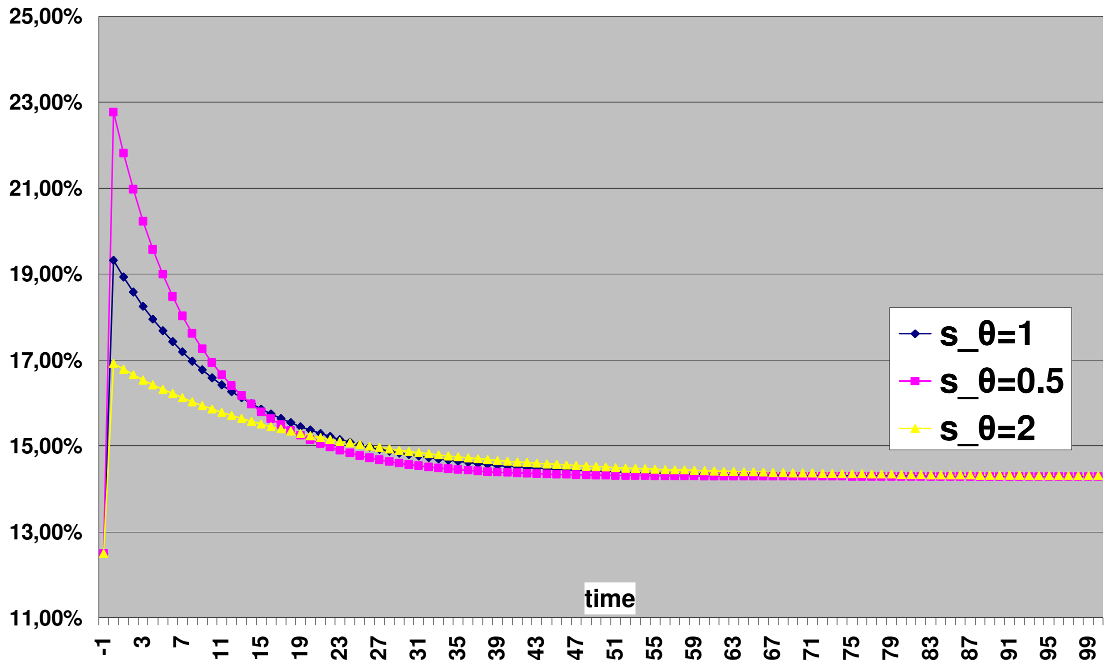

Figure no. 15_A7 The saving rate after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3

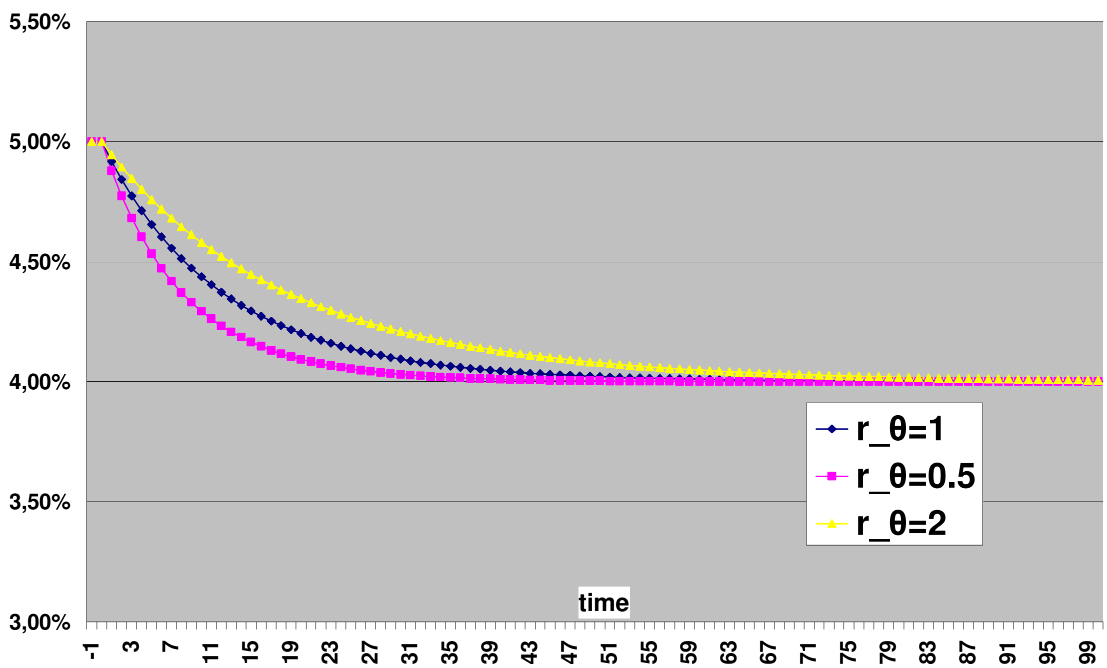

Figure no. 16_A7 The real interest rate after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.

Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3

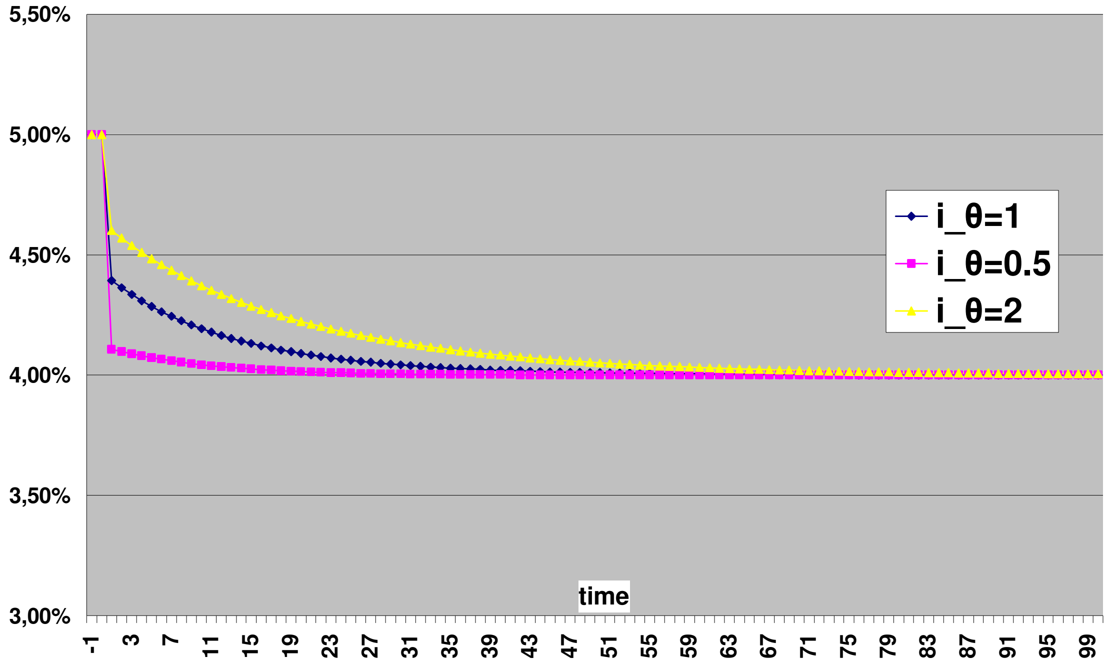

Figure no. 17_A7 The nominal interest rate after the decrease in ρ from 5 % to 4 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/. Money and velocity are constant.

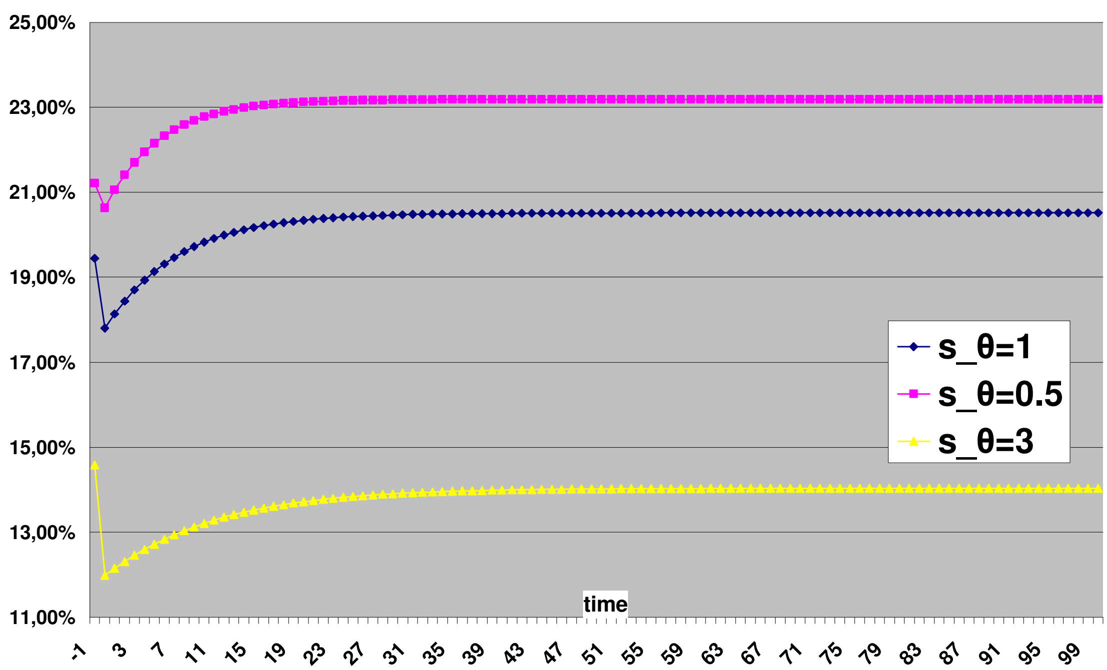

Figure no. 18_A7 The saving rate after the increase in g from 2 % to 3 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3, n = 0 %, ρ = 5 %

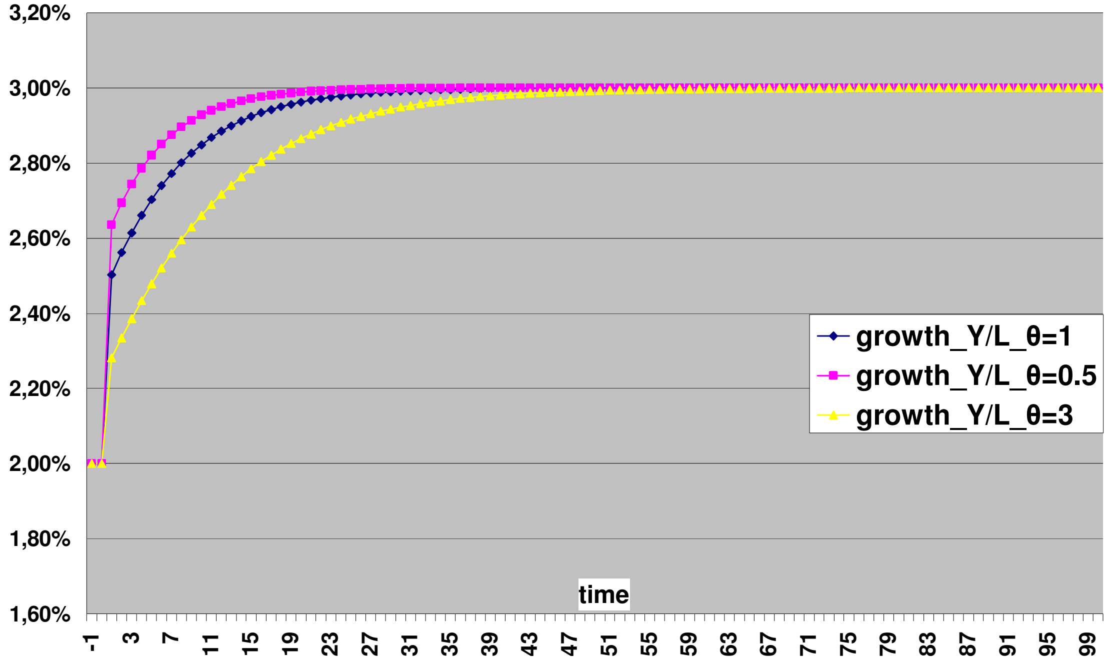

Figure no. 19_A7 The growth rate in GDP per worker after the increase in g from 2 % to 3 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3, n = 0 %, ρ = 5 %

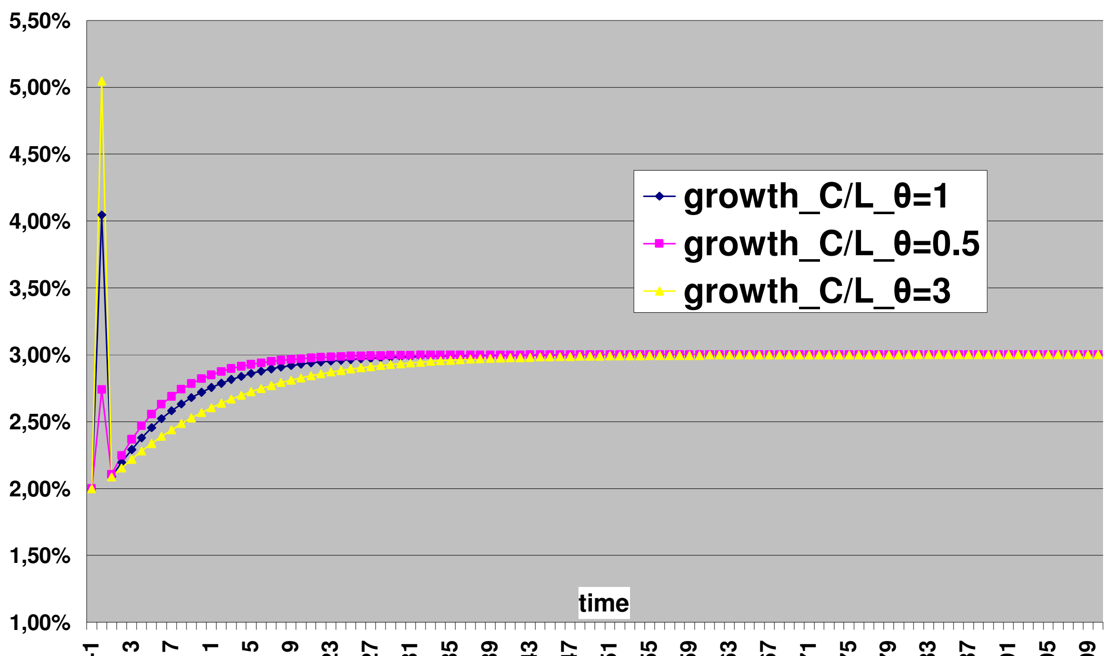

Figure no. 20_A7 The growth rate in consumption per worker after the increase in g from 2 % to 3 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3, n = 0 %, ρ = 5 %

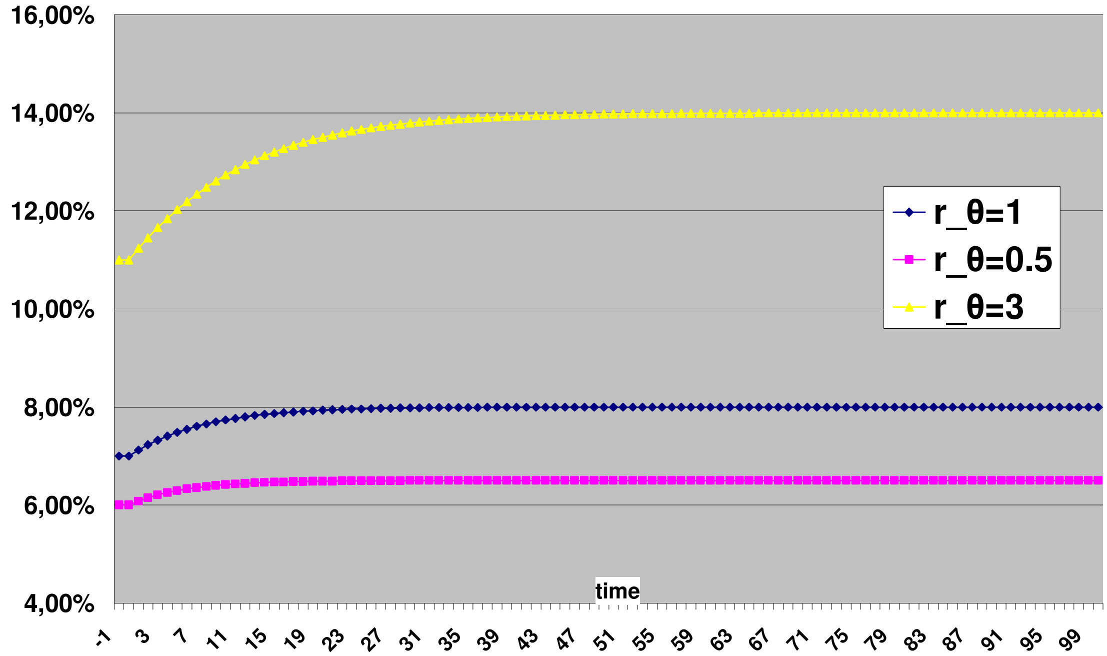

Figure no. 21_A7 The real interest rate after the increase in g from 2 % to 3 % in the RCK model and the role of θ.
Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3, n = 0 %, ρ = 5 %

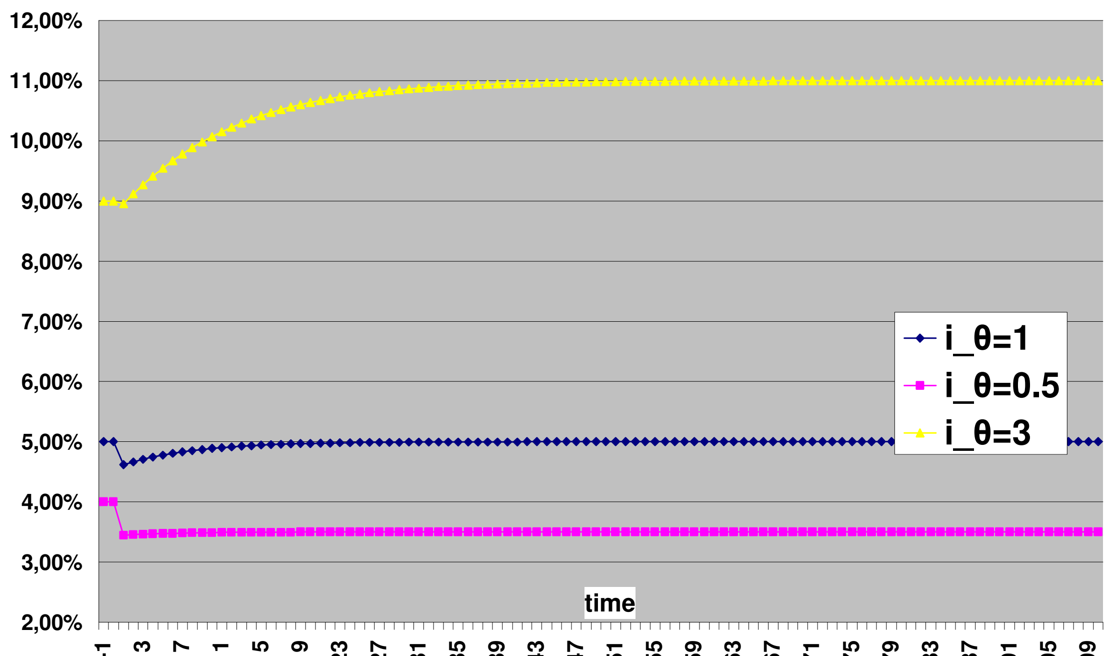

Figure no. 22_A7 The nominal interest rate after the increase in g from 2 % to 3 % in the RCK model and the role of θ.

Note: The set of exogenous parameters is as follows: δ= 3 %, α=1/3, n = 0 %, ρ = 5 %

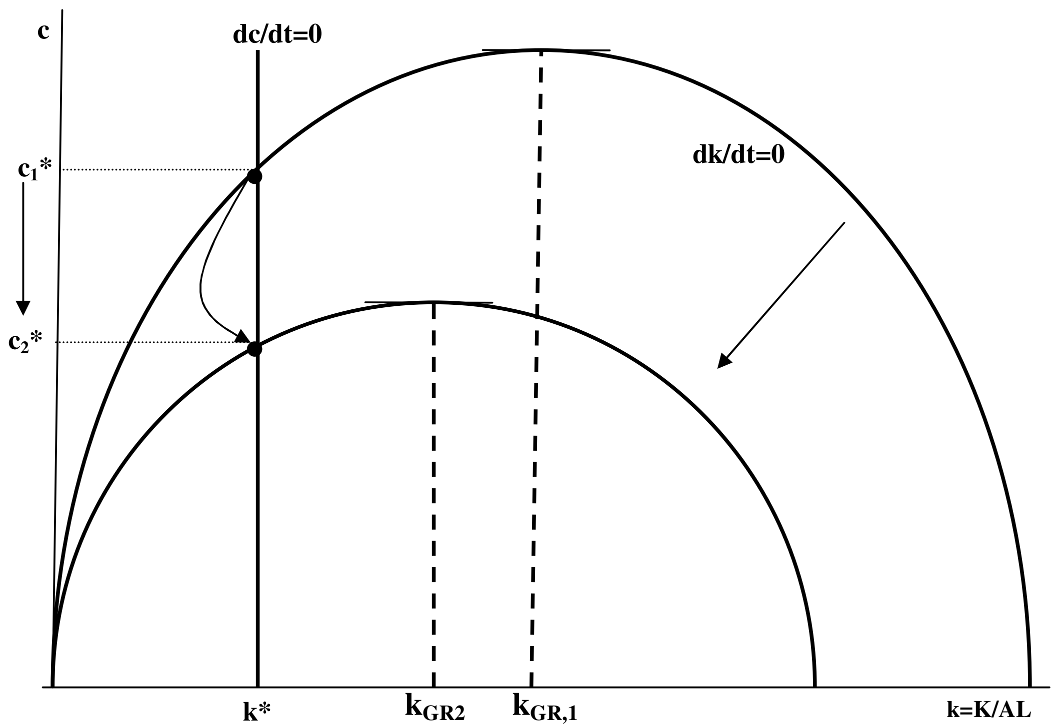

Figure no. 23_A7 Increase in n in the RCK model, if the family is concerned about its offspring.

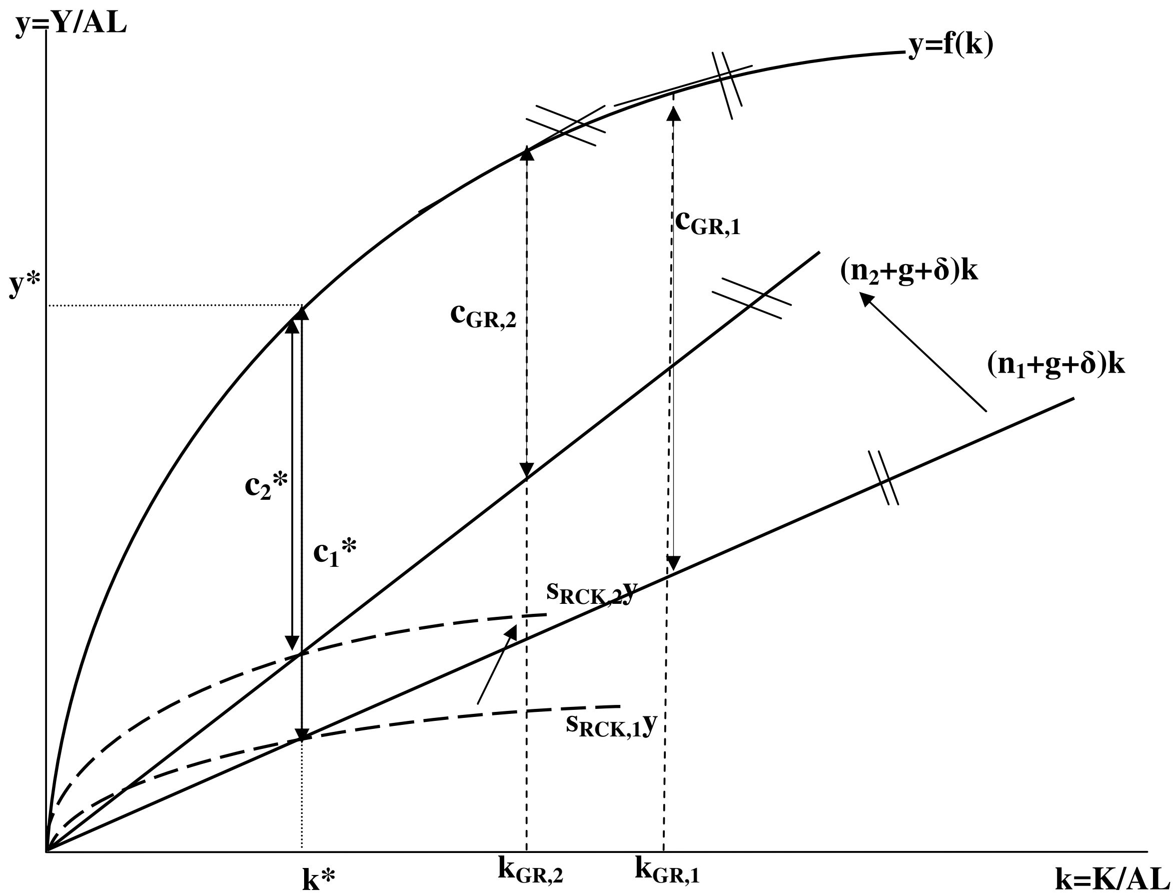

Figure no. 24_A7 Increase in n in the RCK model, if the family is concerned about its offspring; Solow model representation.

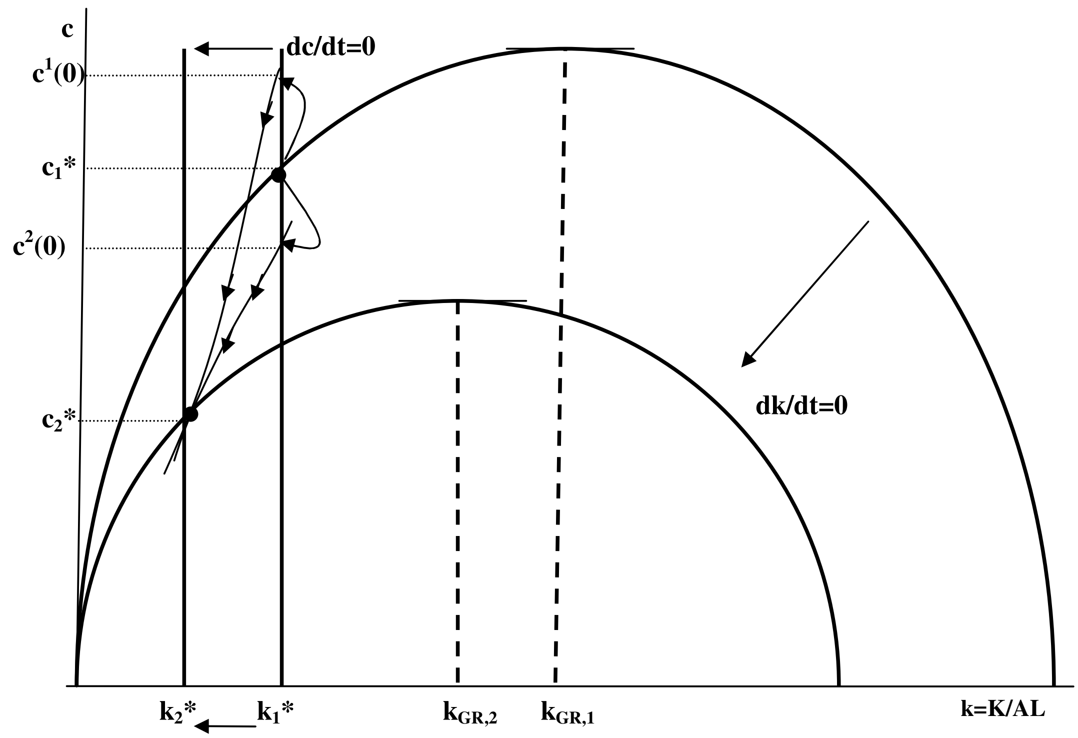

Figure no. 25_A7 Increase in n in the RCK model, the case of "selfish" individuals.
Note: $c^1(0)$ is the initial optimal consumption for low θ, $c^2(0)$ is the initial optimal consumption for high θ.

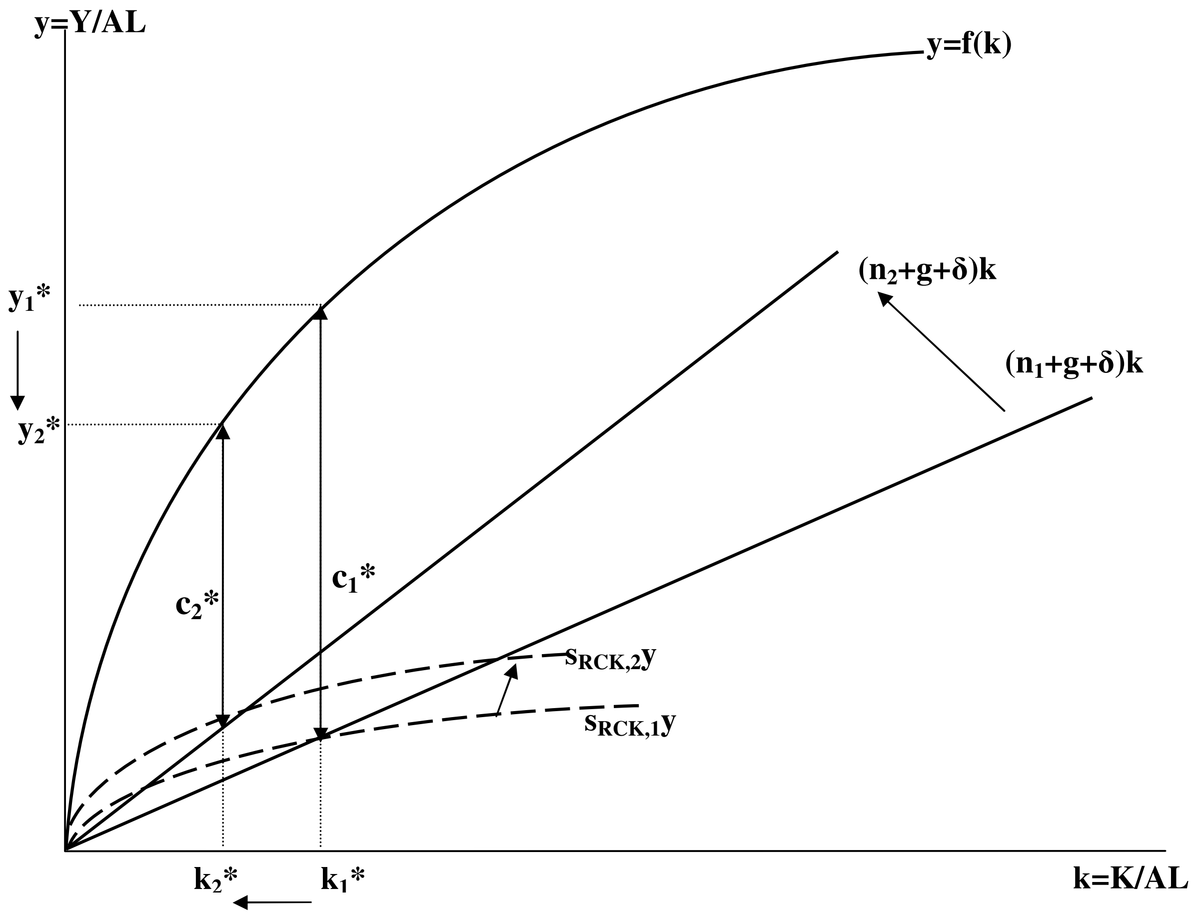

Figure no. 26_A7 Increase in n in the RCK model, the case of “selfish” individuals, Solow model representation.

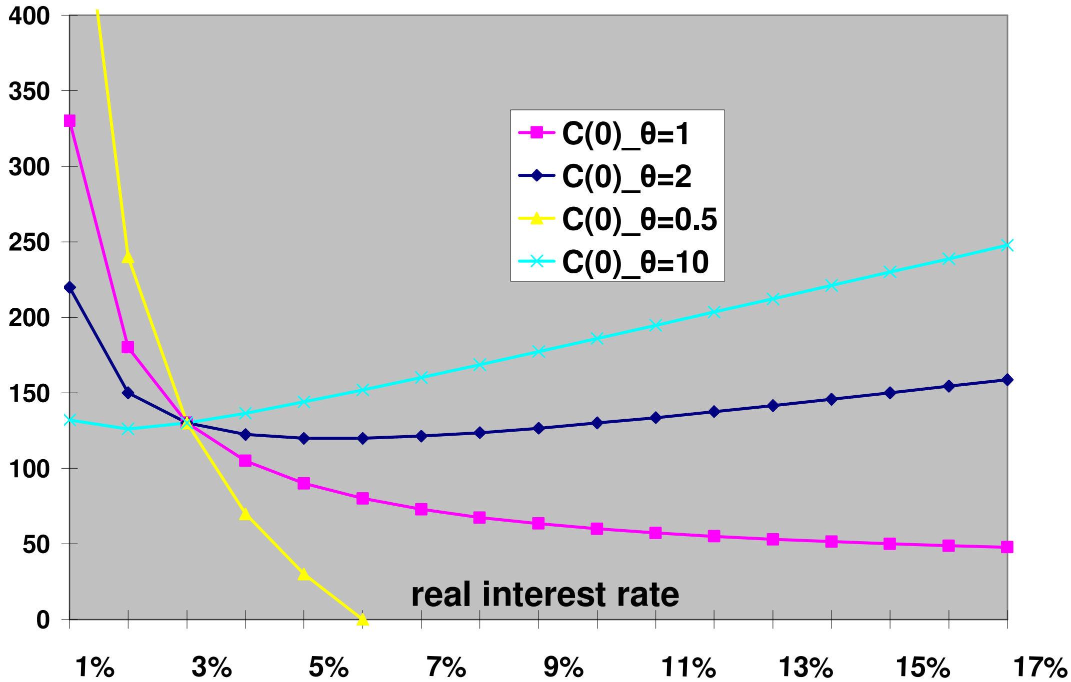

Figure no. 27_A7 The optimum present consumption C(0) for various real interest rates and θ.

Note: The set of exogenous parameters is as follows: W = 100, K(0)/H=1000, ρ = 3 %

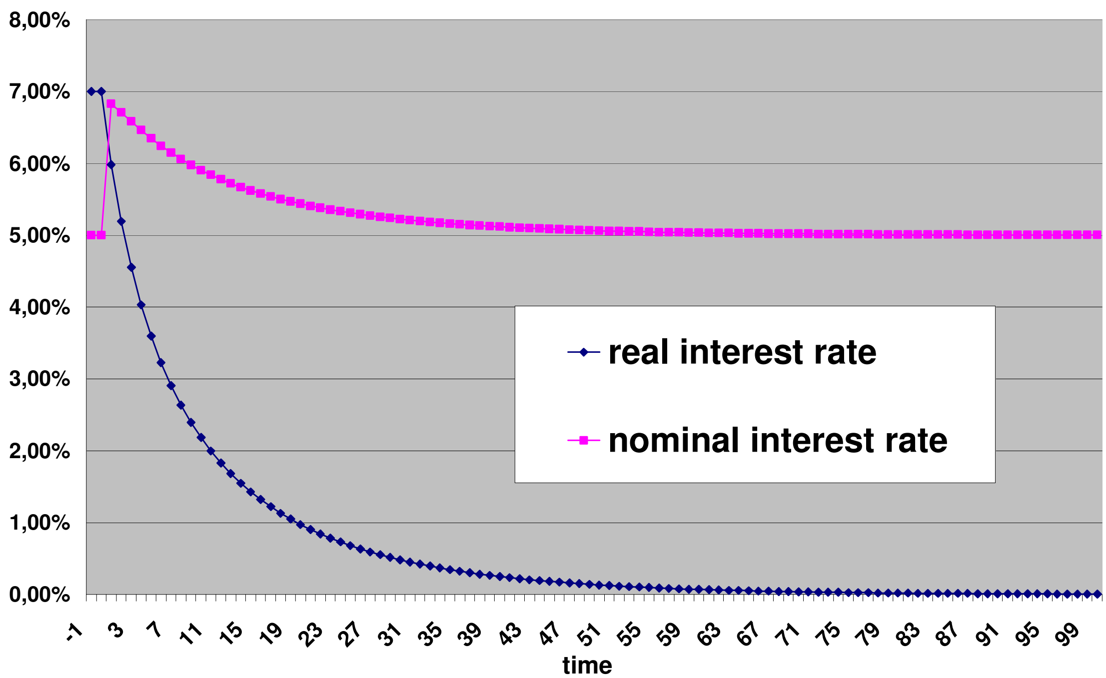

Figure no. 28_A7 The real interest rate gradually approaching zero, if g falls from + 2 % to – 5 %.

Note: The set of exogenous parameters is as follows: δ= 6 %, α=1/3, n = 0 %, ρ = 5 %, θ=1
Nominal interest rate is calculated for constant money and velocity.

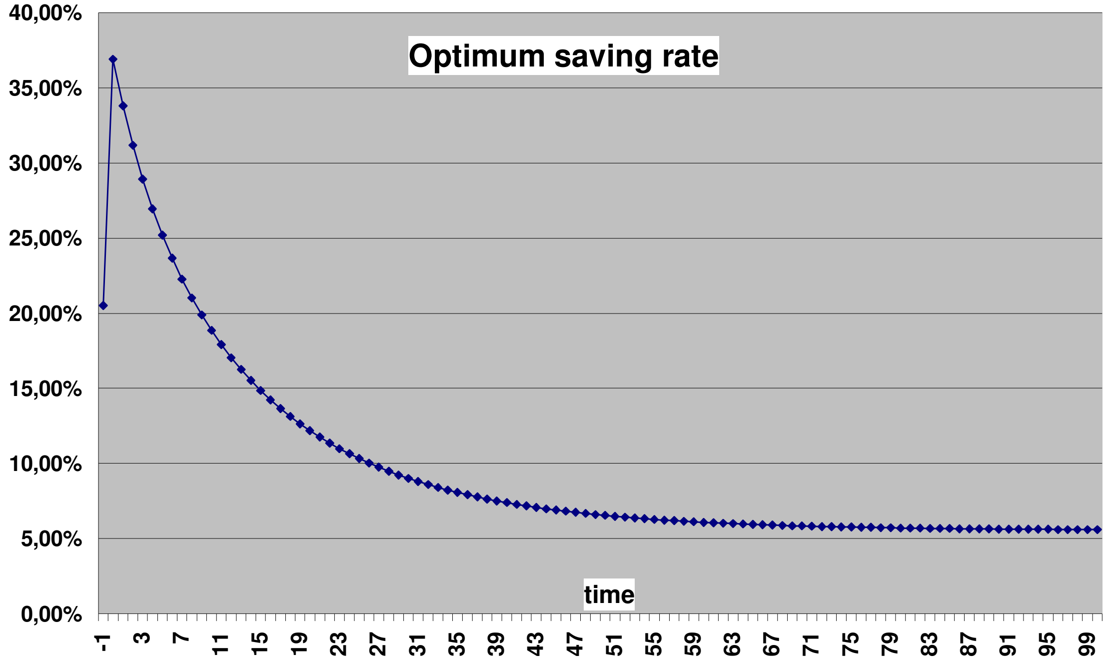

Figure no. 29_A7 The optimum saving rate, if g falls from + 2 % to – 5 %.
Note: The set of exogenous parameters is as follows: δ*= 6 %,* α*=1/3, n = 0 %,* ρ *= 5 %,* θ*=1*

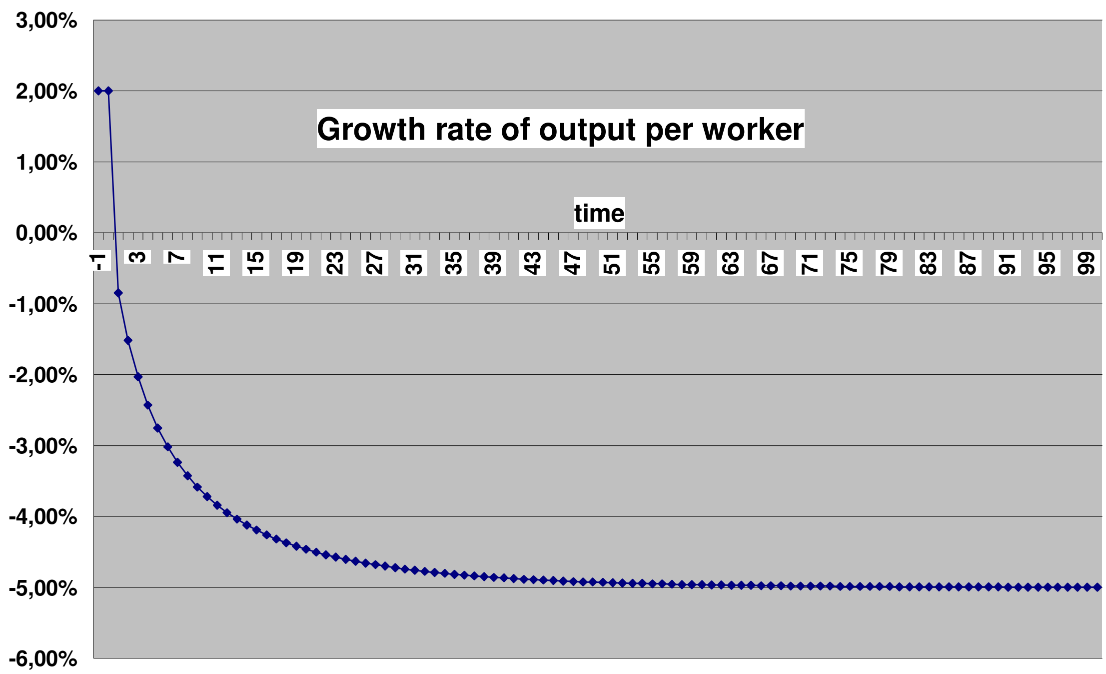

Figure no. 30_A7 The growth rate of output per worker, if g falls from + 2 % to – 5 %.
Note: The set of exogenous parameters is as follows: δ*= 6 %,* α*=1/3, n = 0 %,* ρ *= 5 %,* θ*=1*

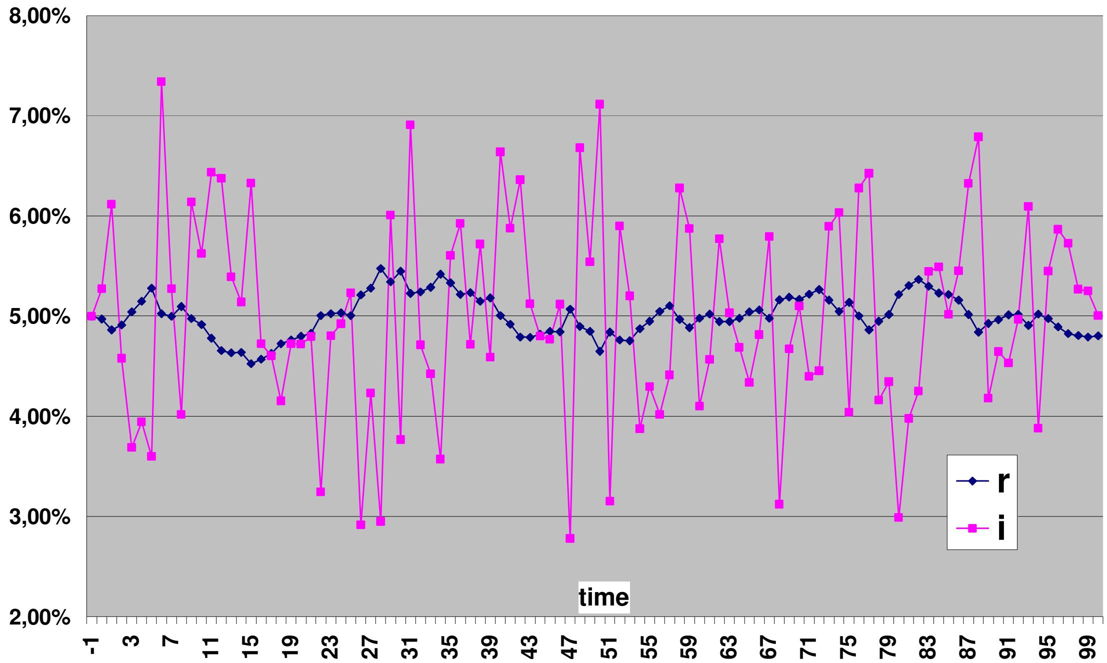

Figure no. 31_A7 Real and nominal interest rate for stochastic A.
Note: The set of exogenous parameters is as follows: δ*= 4 %,* α*=1/3, n = g =0 %,* ρ *= 5 %,* θ*=1. Technological level follows AR(1) process,* β_1*=0.9;* $\varepsilon \sim N(0;\ 0.015^2)$

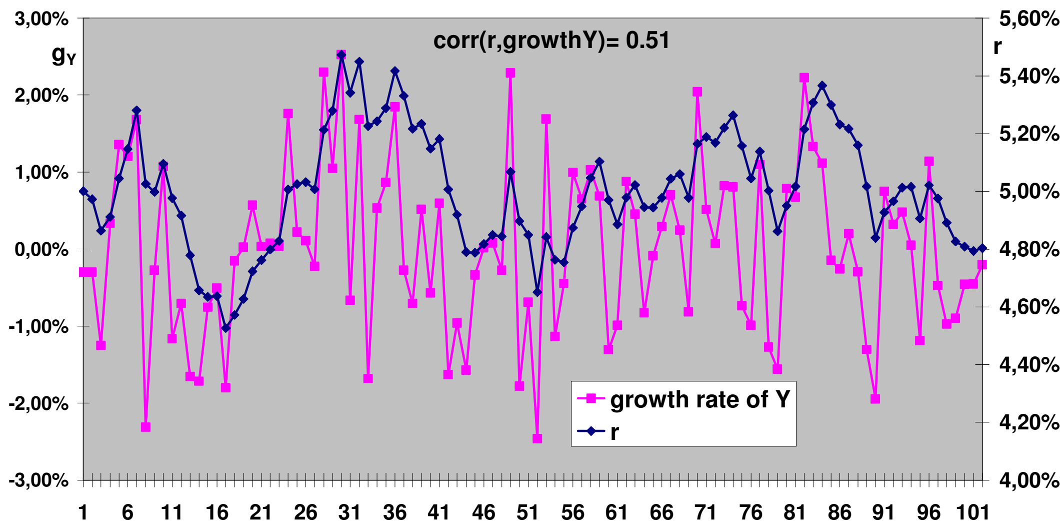

Figure no. 32_A7 Real interest rate and the growth rate in output for stochastic A.
Note: The set of exogenous parameters is as follows: δ*= 4 %,* α*=1/3, n = g =0 %,* ρ *= 5 %,* θ*=1. Technological level follows AR(1) process,* β_1*=0.9;* $\varepsilon \sim N(0;\ 0.015^2)$

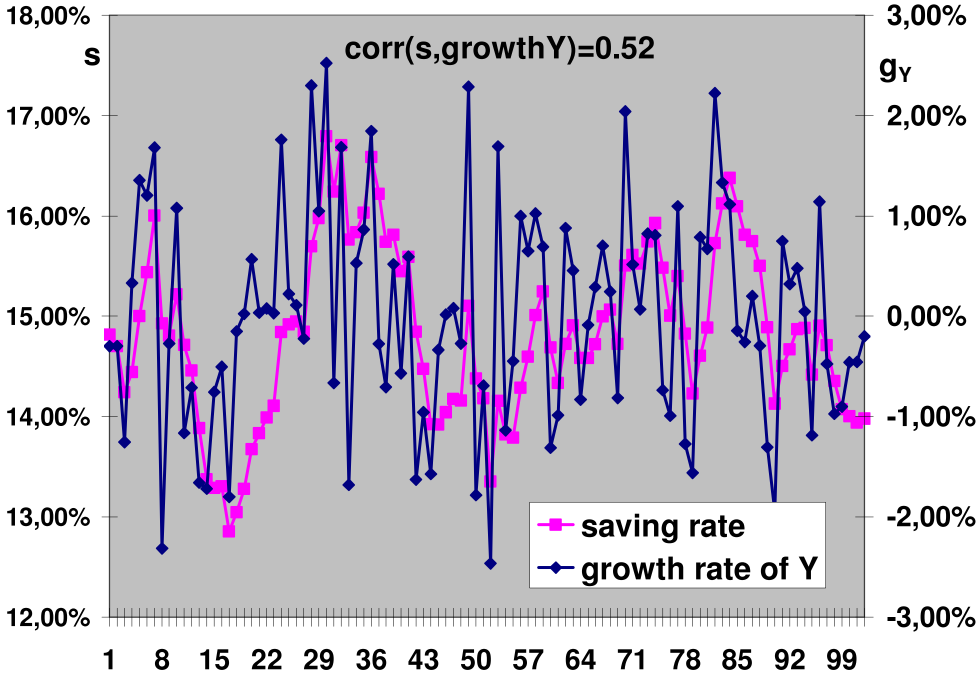

Figure no. 33_A7 Saving rate and the growth rate in output for stochastic A.
Note: The set of exogenous parameters is as follows: δ*= 4 %,* α*=1/3, n = g =0 %,* ρ *= 5 %,* θ*=1. Technological level follows AR(1) process,* β_1*=0.9;* $\varepsilon \sim N(0;\ 0.015^2)$

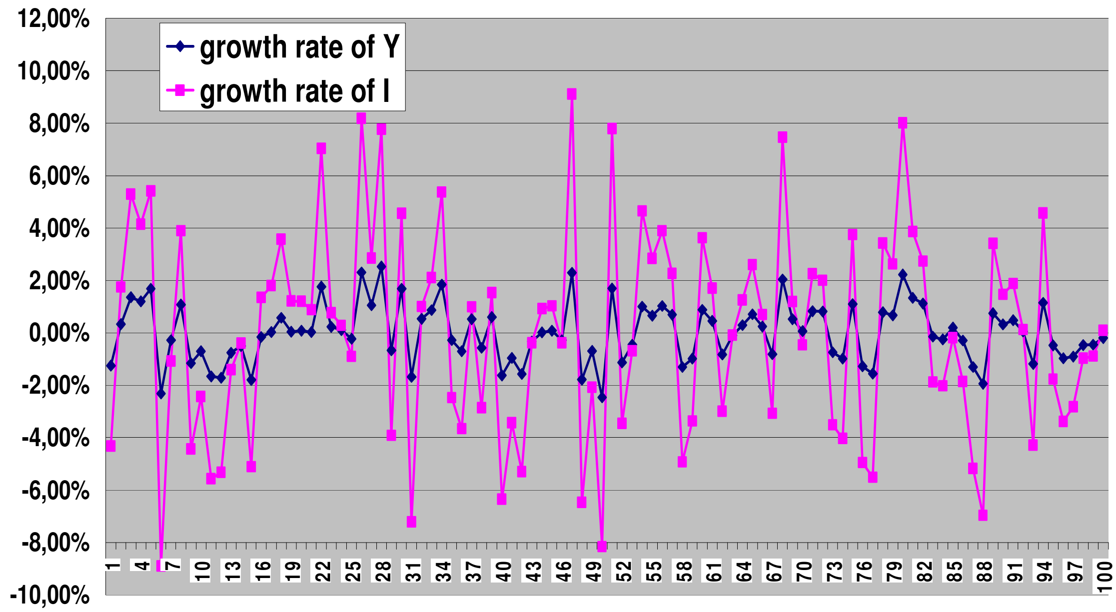

Figure no. 34_A7 Investment growth and the growth rate in output for stochastic A.
Note: The set of exogenous parameters is as follows: δ*= 4 %,* α*=1/3, n = g =0 %,* ρ *= 5 %,* θ*=1. Technological level follows AR(1) process,* β_1*=0.9;* $\varepsilon \sim N(0;\ 0.015^2)$

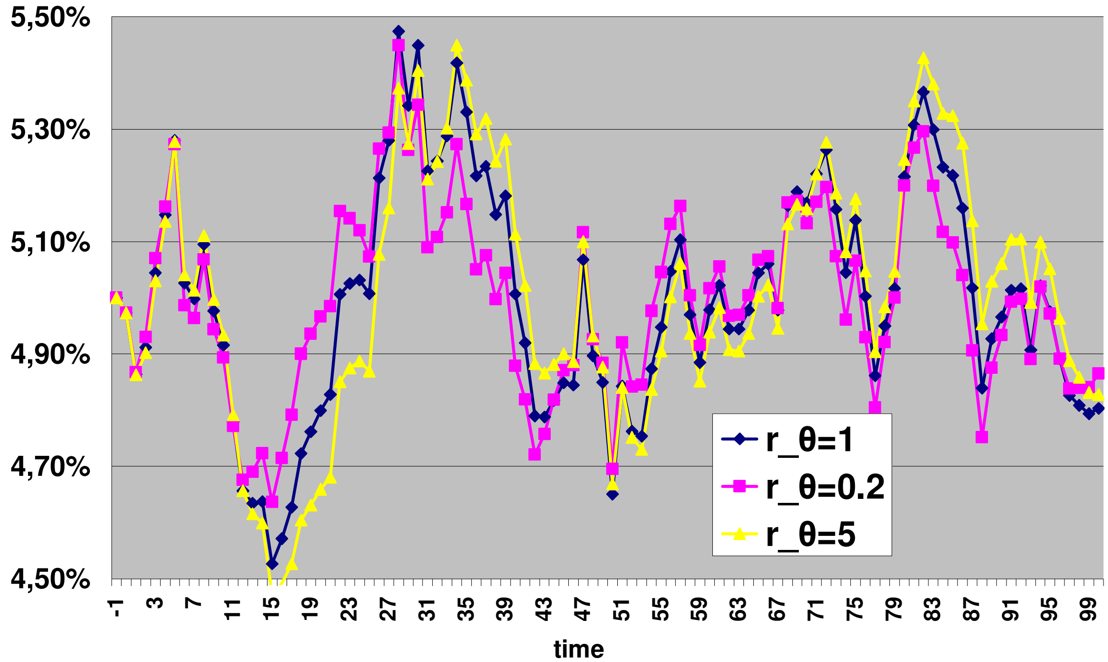

Figure no. 35_A7 Volatility of the natural rate of interest for various θ, if A is stochastic.
Note: The set of exogenous parameters is as follows: $\delta = 4$ %, $\alpha = 1/3$, $n = g = 0$ %, $\rho = 5$ %, Technological level follows AR(1) process, $\beta_1 = 0.9$; $\varepsilon \sim N(0;\ 0.015^2)$

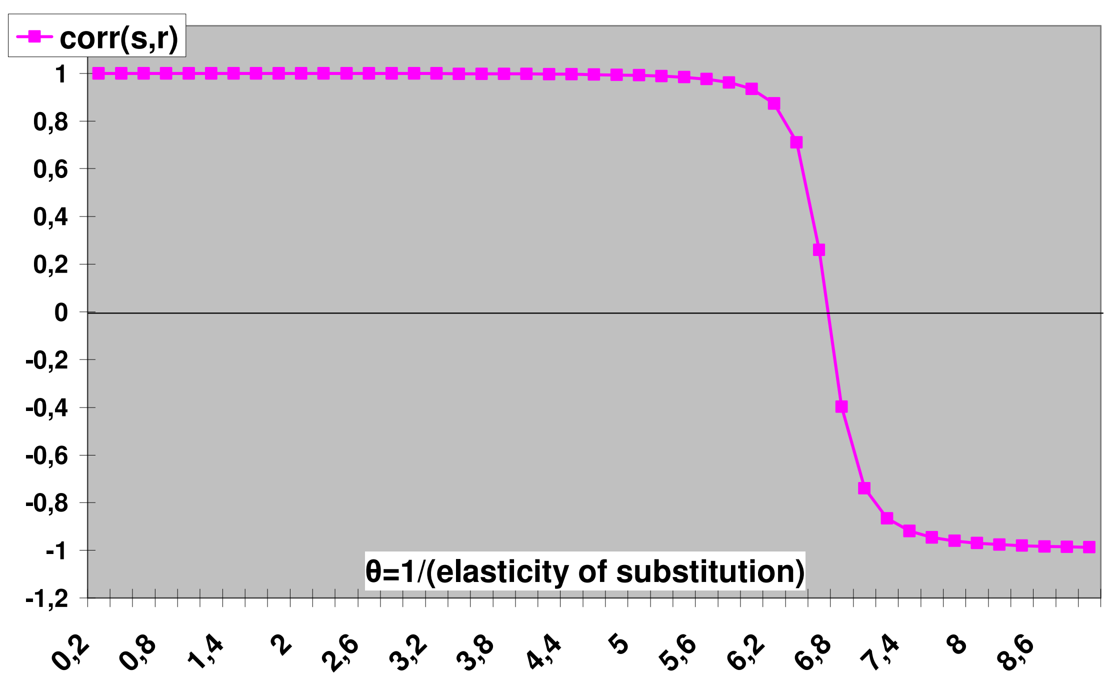

Figure no. 35_A7B Correlation between the natural rate of interest and the saving rate for various θ, if A is stochastic.
Note: The set of exogenous parameters is as follows: $\delta = 4$ %, $\alpha = 1/3$, $n = g = 0$ %, $\rho = 5$ %, Technological level follows AR(1) process, $\beta_1 = 0.9$; $\varepsilon \sim N(0;\ 0.015^2)$

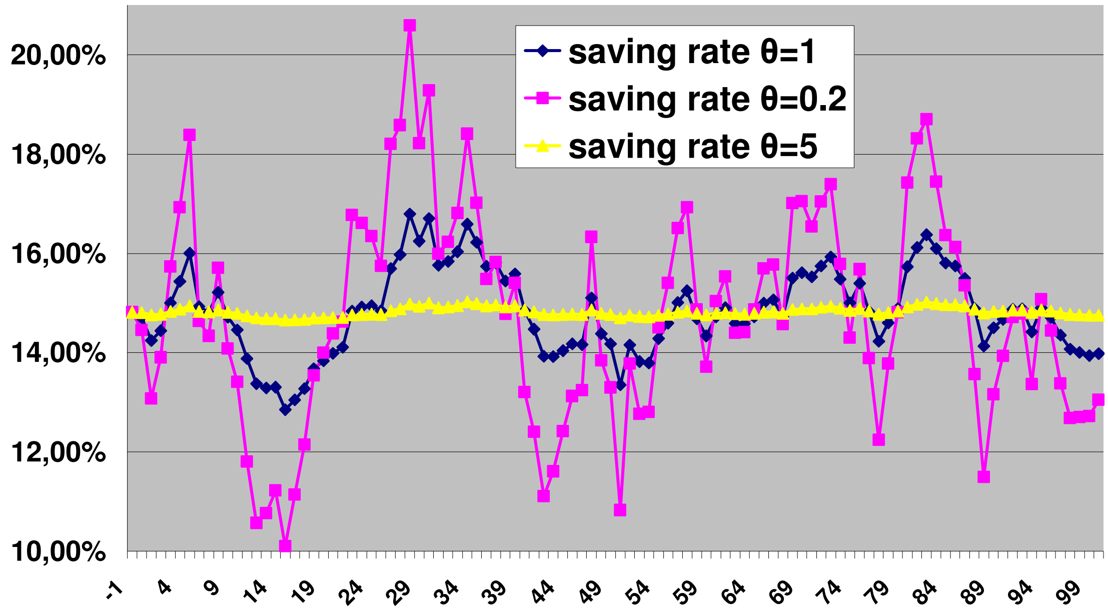

Figure no. 36_A7 Volatility of the saving rate for various θ, if A is stochastic.
Note: The set of exogenous parameters is as follows: δ= 4 %, α=1/3, n = g =0 %, ρ = 5 %, Technological level follows AR(1) process, β_1*=0.9; ε~N(0; 0.015*^{2}*)*

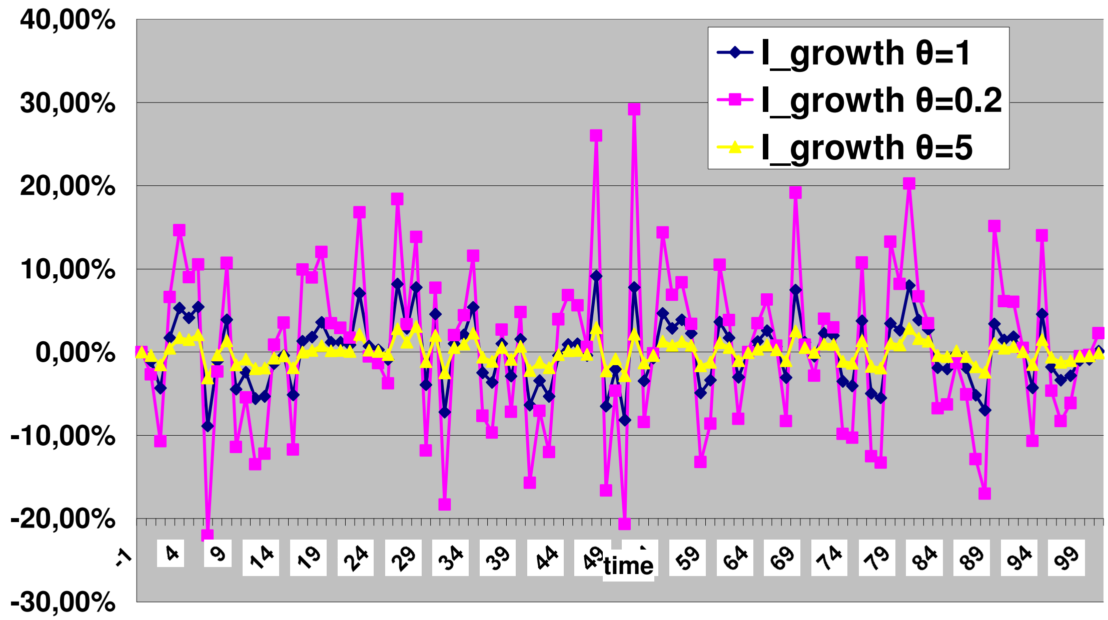

Figure no. 37_A7 Volatility of investment growth for various θ, if A is stochastic.
Note: The set of exogenous parameters is as follows: δ= 4 %, α=1/3, n = g =0 %, ρ = 5 %, Technological level follows AR(1) process, β_1*=0.9; ε~N(0; 0.015*^{2}*)*

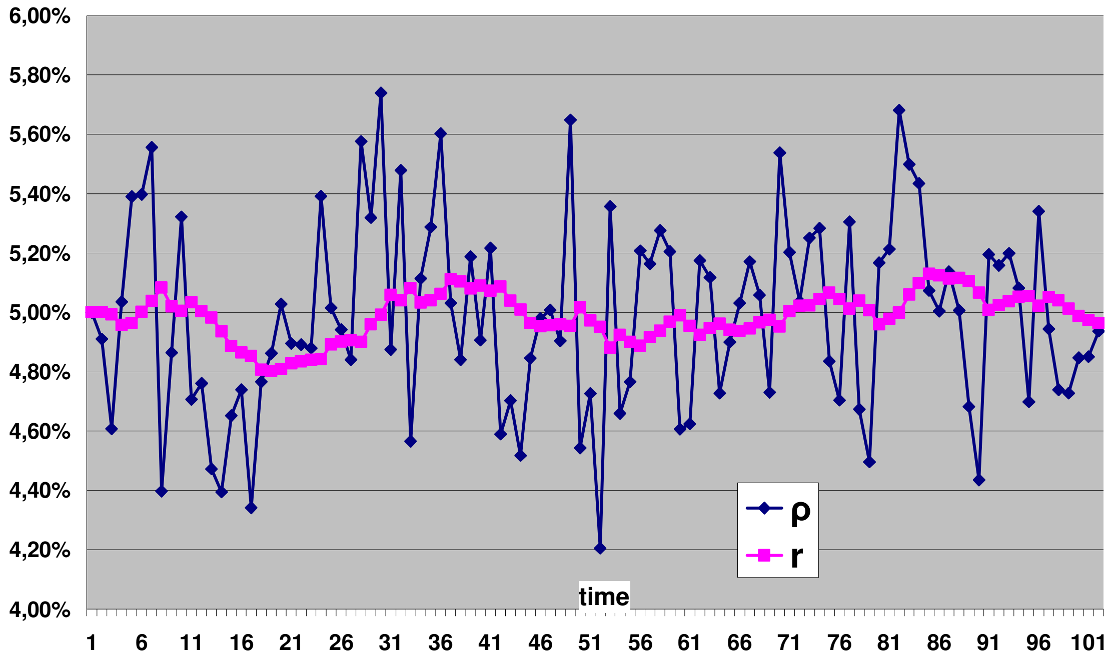

Figure no. 38_A7 Volatility of the natural rate of interest, if ρ is stochastic.
Note: The set of exogenous parameters is as follows: δ= 4 %, α=1/3, n = g =0 %, θ=1, subjective discount rate follows an MA(1) process, γ=0.1; $\mu \sim N(0; 0.003^2)$

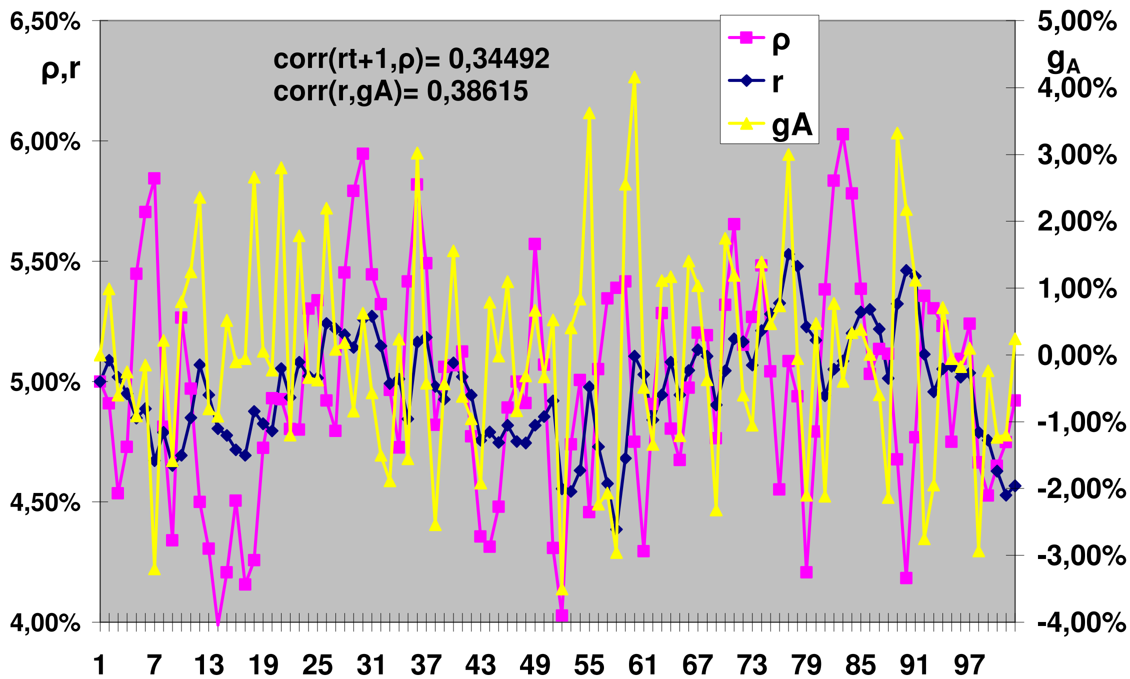

Figure no. 39_A7 Volatility of the natural rate of interest, if both ρ and A are stochastic.
Note: The set of exogenous parameters is as follows: δ= 4 %, α=1/3, n = g =0 %, θ=1, subjective discount rate follows an MA(1) process, γ=0.1; $\mu \sim N(0; 0.003^2)$. Technological level follows AR(1) process, β_1=0.9; $\varepsilon \sim N(0; 0.015^2)$