Tutorials in Microeconomics

Ivo Koubek[†] et al.[*]

2022

[*]Petr Maleček, Martin Janíčko, Pavel Potužák

Contents

Preface

The *Tutorials in Microeconomics* is an exercise book intended for students—be it bachelors or masters—who are interested in microeconomics and/or have to pass various University courses in this field. The main objective of the book is to offer a set of exercises that will enable the students to test and master their knowledge on examples with varying difficulty. Most of the computational problems included in the book are up-to-date and also have a realistic background.

The examples have been accumulated over the years of teaching microeconomics at top Czech schools of economics, meet the needs of students for the final exams and at the same time expand their ability to understand and apply microeconomic principles in real life.

The authors believe that by mastering them, the users of this exercise book will gain additional affection for such a beautiful discipline as microeconomics and at the same time practice analytical thinking while solving hopefully interesting assignments.

The book also carries the legacy of Ivo Koubek, its main author, who passed away in 2017. Ivo Koubek was not only an excellent microeconomist, but what was unique was his commitment to making sure that the examples and exercises he presented to students were understandable, despite their potential technical complexity, while at the same time being as close to reality as possible, and also that they came out in integer form.

Any constructive feedback from the users of this exercise book is of course much appreciated and welcome.

Authors
Ivo Koubek (main author, *in memoriam*)
Petr Maleček (petrxmalecek [at] gmail.com)
Martin Janíčko (martin.janicko [at] email.cz)
Pavel Potužák (pavel.potuzak [at] vse.cz)

Chapter 1

Optimization

Problem 1 *Find possible extreme points of the function $z = xy$, subject to the constraint $x + y = 6$.*

Solution. The Lagrange function has the following form:

$$L(x, y, \lambda) = xy + \lambda [6 - x - y].$$

First-order conditions (FOC):

$$\frac{\partial L}{\partial x} = y - \lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial y} = x - \lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial \lambda} = 6 - x - y \stackrel{!}{=} 0.$$

It follows from the first two equations that $x = y$, after substituting into the third one we get $x^* = y^* = \lambda^* = 3$. ■

Problem 2 *Find possible extreme points which of the function $z = x^2 + y^2$ subject to the constraint $x + 4y = 2$.*

Solution. The Lagrange function has the following form:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda [2 - x - 4y].$$

First-order conditions (FOC) are as follows:

$$\frac{\partial L}{\partial x} = 2x - \lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial y} = 2y - 4\lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial \lambda} = 2 - x - 4y \stackrel{!}{=} 0.$$

From the first two equations we obtain $y = 4x$, so that $x^* = \frac{2}{17}$, $y^* = \frac{8}{17}$ and $\lambda^* = \frac{4}{17}$. ■

Problem 3 *Solve the following problem.*

$$\begin{aligned} & \min \ z = (x_1-4)^2 + (x_2-4)^2 \\ s.t. \quad & 2x_1 + 3x_2 \geq 6 \\ & -3x_1 - 2x_2 \geq -12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution. The Lagrange function has the following form:

$$L(x,\lambda) = (x_1-4)^2 + (x_2-4)^2 + \lambda_1 [6 - 2x_2 - 3x_2] + \lambda_2 [-12 + 3x_1 + 2x_2].$$

The Kuhn-Tucker sufficient conditions for the minimum:

$$\frac{\partial L}{\partial x_1} = 2(x_1-4) - 2\lambda_1 + 3\lambda_2 \geq 0, \quad [2(x_1-4) - 2\lambda_1 + 3\lambda_2]\, x_1 = 0, \tag{1.1}$$

$$\frac{\partial L}{\partial x_2} = 2(x_2-4) - 3\lambda_1 + 2\lambda_2 \geq 0, \quad [2(x_2-4) - 3\lambda_1 + 2\lambda_2]\, x_2 = 0, \tag{1.2}$$

$$\frac{\partial L}{\partial \lambda_1} = 6 - 2x_1 - 3x_2 \leq 0, \quad [6 - 2x_1 - 3x_2]\, \lambda_1 = 0, \tag{1.3}$$

$$\frac{\partial L}{\partial \lambda_2} = -12 + 3x_1 + 2x_2 \leq 0, \quad [-12 + 3x_1 + 2x_2]\, \lambda_2 = 0, \tag{1.4}$$

including the non-negativity conditions for $x_1, x_2, \lambda_1, \lambda_2 \geq 0$. Let us try to put $x_1 > 0, x_2 > 0, \lambda_1 = 0$ and $\lambda_2 = 0$. It follows from the complementarity conditions (1.1) a (1.2) that $x_1 = x_2 = 4$. However, we need to verify if all conditions are met, including (1.3) and (1.4). After inserting into (1.4) we reach a contradiction, since

$$\left.\frac{\partial L}{\partial \lambda_2}\right|_{x_1=x_2=4} = 8 \nleq 0,$$

so that the point $[x_1, x_2, \lambda_1, \lambda_2] = [4, 4, 0, 0]$ cannot be a maximum. Let us now try to put $x_1 > 0, x_2 > 0, \lambda_1 = 0$ and $\lambda_2 > 0$. From the complementarity conditions we obtain the following set of equations

$$\begin{aligned} 2x_1 \qquad + 3\lambda_2 &= 8, \\ 2x_2 + 2\lambda_2 &= 8, \\ 3x_1 \qquad + 2x_2 &= 12, \end{aligned}$$

with a solution $[x_1, x_2, \lambda_2] = \left[\frac{28}{13}, \frac{36}{13}, \frac{16}{13}\right]$. We can easily verify that the solution $[x_1, x_2, \lambda_1, \lambda_2] = \left[\frac{28}{13}, \frac{36}{13}, 0, \frac{16}{13}\right]$ also satisfies the remaining Kuhn-Tucker conditions, and is therefore the sought-after maximum of the problem. ■

Problem 4 *Solve the following problem.*

$$\begin{aligned} \max\ z &= 2x_1 + x_2 \\ s.t. \quad -x_1^2 + 4x_1 - x_2 &\leq 0, \\ 2x_1 + 3x_2 &\leq 12, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution. The Lagrange function has the following form:

$$L(x, \lambda) = 2x_1 + x_2 + \lambda_1 \left[x_1^2 + 4x_1 + x_2\right] + \lambda_2 \left[12 - 2x_1 - 3x_2\right].$$

Sufficient conditions for the maximum are as follows:

$$\frac{\partial L}{\partial x_1} = 2 + 2\lambda_1 x_1 + 4\lambda_1 - 2\lambda_2 \leq 0, [2 + 2\lambda_1 x_1 + 4\lambda_1 - 2\lambda_2]\, x_1 = 0, \tag{1.5}$$

$$\frac{\partial L}{\partial x_2} = 1 + \lambda_1 - 3\lambda_2 \leq 0, \quad [1 + \lambda_1 - 3\lambda_2]\, x_2 = 0, \tag{1.6}$$

$$\frac{\partial L}{\partial \lambda_1} = x_1^2 + 4x_1 + x_2 \geq 0, \quad \left[x_1^2 + 4x_1 + x_2\right] \lambda_1 = 0, \tag{1.7}$$

$$\frac{\partial L}{\partial \lambda_2} = 12 - 2x_1 - 3x_1 \geq 0, \quad [12 - 2x_1 - 3x_1]\, \lambda_2 = 0, \tag{1.8}$$

including the non-negativity conditions for $x_1, x_2, \lambda_1, \lambda_2 \geq 0$. Let us try to set $x_1 > 0, x_2 > 0, \lambda_1 > 0, \lambda_2 > 0$. From the complementarity conditions we obtain the following set of equations

$$\begin{aligned} 2 + 2\lambda_1 x_1 + 4\lambda_1 - 2\lambda_2 &= 0, \\ 1 + \lambda_1 - 3\lambda_2 &= 0, \\ x_1^2 + 4x_1 + x_2 &= 0, \\ 12 - 2x_1 - 3x_2 &= 0. \end{aligned}$$

If we solve for x_2 in the fourth equation and put it into the third, after some manipulation we obtain the quadratic equation $3x_1^2 + 10x_1 + 12 = 0$ with a negative discriminant. Therefore, we reach a contradiction with the assumption that $x_1 \in \mathbb{R}$. Let us try to put $x_1 > 0, x_2 > 0, \lambda_1 > 0, \lambda_2 = 0$. From the complementarity condition (1.6) we see immediately that neither this assumption is correct since it implies that $\lambda_1 = -1 \ngeq 0$. Let us try further $x_1 > 0, x_2 > 0, \lambda_1 = 0, \lambda_2 > 0$. From the complementarity assumption (1.5) we get $\lambda_2 = 1$, whereas (1.6) implies $\lambda_2 = 1/3$, which is a contradiction. Let us put $x_1 = 0, x_2 > 0, \lambda_1 = 0, \lambda_2 > 0$. It follows from the equations (1.6) and (1.8) that $x_2 = 4$ and $\lambda_2 = 1/3$. We can easily verify that the solution $[x_1, x_2, \lambda_1, \lambda_2] = \left[0, 4, 0, \frac{1}{3}\right]$ satisfies also the remaining Kuhn-Tucker conditions and is therefore the sought-after maximum of the problem. ■

Problem 5 *Solve the following problem.*

$$\begin{aligned} \min\ z &= x_1^2 + x_2^2 \\ s.t.\ \ x_1 x_2 &\geq 25, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution. The Lagrange function has the following form:

$$L(x, \lambda) = x_1^2 + x_2^2 + \lambda \left[25 - x_1 x_2\right].$$

Sufficient conditions for the minimum:

$$\frac{\partial L}{\partial x_1} = 2x_1 - \lambda x_2 \geq 0, \quad \left[2x_1 - \lambda x_2\right] x_1 = 0, \tag{1.9}$$

$$\frac{\partial L}{\partial x_2} = 2x_2 - \lambda x_1 \geq 0, \quad \left[2x_2 - \lambda x_1\right] x_2 = 0, \tag{1.10}$$

$$\frac{\partial L}{\partial \lambda} = 25 - x_1 x_2 \leq 0, \quad \left[25 - x_1 x_2\right] \lambda = 0, \tag{1.11}$$

including non-negativity conditions $x_1, x_2, \lambda \geq 0$. From the condition (1.11) we can observe that both x_1 and x_2 must be positive. This further implies that $\lambda > 0$, see equations (1.9) and (1.10). We then need to solve the following set of equations

$$\begin{aligned} 2x_1 - \lambda x_2 &= 0, \\ 2x_2 - \lambda x_1 &= 0, \\ 25 - x_1 x_2 &= 0. \end{aligned}$$

Its solution is the point $[x_1, x_2, \lambda] = [5, 5, 2]$. ■

Problem 6 *We assume the expenditure minimisation function (as per the Hicksian demand). Find a minimum of the following function: $z = x^2 + y^2$ with respect to a constraint given as $x + 4y = 2$*

Solution. The Lagrange function has the following form:

$$L(x, y, \lambda) = x^2 + y^2 + \lambda \left[2 - x - 4y\right].$$

First-order conditions (FOC) are obtained by solving three equations with three unknowns:

$$\frac{\partial L}{\partial x} = 2x - \lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial y} = 2y - 4\lambda \stackrel{!}{=} 0, \quad \frac{\partial L}{\partial \lambda} = 2 - x - 4y \stackrel{!}{=} 0.$$

From the first two equations we get $y = 4x$, so that $x^* = \frac{2}{17}$, $y^* = \frac{8}{17}$ a $\lambda^* = \frac{4}{17}$.

Second-order conditions (SOC) are obtained by solving bordered Hessian in the following form:

$$\text{Hess} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & -4 \\ -1 & -4 & 0 \end{vmatrix}$$

$$= (0+0+0) - (2+32+0) = -34 < 0. \tag{1.12}$$

Hence, we can see that this is indeed a minimum. This could be also recognised from the fact that the function is quasiconvex (2,0 and 0,2 in the first two rows and columns of the matrix combined with the linear constraint defined initially). ■

Exercise 1 *Solve the following problem:*

$$\begin{aligned} \max\ z &= x_1^2 + (x_2-2)^2 \\ s.t.\ \ 5x_1 + 3x_2 &\leq 15, \\ x_1, x_2 &\geq 0. \end{aligned}$$

$[x_1^* = 4, x_2^* = 7]$

Exercise 2 *Solve the following problem:*

$$\begin{aligned} \min\ z &= x_1 + x_2 \\ s.t.\ \ x_1^2 + x_2 &\geq 9, \\ -x_1x_2 &\geq -8 \\ x_1, x_2 &\geq 0. \end{aligned}$$

$[x_1^* = 4, x_2^* = 7]$

Exercise 3 *Solve the following problem:*

$$\begin{aligned} \min\ z &= (x_1-4)^2 + (x_2-4)^2 \\ s.t.\ \ 2x_1 + 3x_2 &\geq 6, \\ -3x_1 - 2x_2 &\geq -12 \\ x_1, x_2 &\geq 0. \end{aligned}$$

$\left[x_1^* = 2\frac{2}{13}, x_2^* = 2\frac{10}{13}\right]$

Chapter 2

Consumer Theory

Problem 7 *Let us assume a consumer with a Cobb-Douglas utility function in the following form:*

$$u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha},\ \alpha \in (0,1).$$

Derive the Marshallian demands $x_1^* = D_1(p_1, p_2, M)$ *and* $x_2^* = D_1(p_1, p_2, M)$.

Solution. Since preferences are monotonous, budget constraint will be satisfied as an equality. Notice further that indifference curves of the Cobb-Douglas utility function do not allow for corner solution. Therefore, the Kuhn-Tucker conditions for a maximum are the same as first order Lagrange conditions. The Lagrange function has the following form:

$$L(x_1, x_2, \lambda) = x_1^\alpha x_2^{1-\alpha} + \lambda \left[M - p_1 x_1 - p_2 x_2\right].$$

First-order conditions (FOC) for an extreme:

$$\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 \overset{!}{=} 0, \tag{2.1}$$

$$\frac{\partial L}{\partial x_2} = (1-\alpha) x_1^\alpha x_2^{-\alpha} - \lambda p_2 \overset{!}{=} 0, \tag{2.2}$$

$$\frac{\partial L}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 \overset{!}{=} 0. \tag{2.3}$$

It follows from the equations (2.1) and (2.2) that

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2},$$

so that it holds for x_2 that

$$x_2 = \frac{(1-\alpha)}{\alpha} \frac{p_1}{p_2} x_1.$$

If we insert this expression into (2.3) and solve for x_1, we obtain

$$x_1^* = D_1(p_1, p_2, M) = \frac{aM}{p_1}.$$

From the formulation for x_2 it then follows that

$$x_2^* = D_2(p_1, p_2, M) = \frac{(1-a)M}{p_2}.$$

■

Problem 8 *Assume a consumer with a utility function exhibiting a constant elasticity of substitution*

$$u(x_1, x_2) = A(\delta_1 z_1^\alpha + \delta_2 z_2^\alpha)^{1/\alpha}, \quad \delta_1 + \delta_2 = 1, \quad A > 0.$$

Derive the Marshallian demand functions $x_1^* = D_1(p_1, p_2, M)$ *a* $x_2^* = D_1(p_1, p_2, M)$*,* p_1 *and* p_2 *representing prices and* M *income.*

Solution. It follows from the monotony of preferences that budget constraint will be satisfied as an equality. We will assume an interior solution and therefore use the Lagrange multipliers. Lagrange function has the following form:

$$L(x_1, x_2, \lambda) = A(\delta_1 x_1^\alpha + \delta_2 x_2^\alpha)^{1/\alpha} + \lambda\left[M - p_1 x_1 - p_2 x_2\right].$$

First-order conditions (FOC) for an extreme:

$$\frac{\partial L}{\partial x_1} = \frac{1}{\alpha} A(\delta_1 x_1^\alpha + \delta_2 x_2^\alpha)^{\frac{1}{\alpha}-1}(\alpha\delta_1 x_1^{\alpha-1}) - \lambda p_1 \stackrel{!}{=} 0, \tag{2.4}$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{\alpha} A(\delta_1 x_1^\alpha + \delta_2 x_2^\alpha)^{\frac{1}{\alpha}-1}(\alpha\delta_2 x_2^{\alpha-1}) - \lambda p_2 \stackrel{!}{=} 0, \tag{2.5}$$

$$\frac{\partial L}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 \stackrel{!}{=} 0. \tag{2.6}$$

It follows from (2.4) a (2.5) that

$$\frac{\delta_1}{\delta_2}\left(\frac{x_1}{x_2}\right)^{\alpha-1} = \frac{p_1}{p_2},$$

so that

$$x_1 = x_2\left(\frac{p_1\delta_2}{p_2\delta_1}\right)^{\frac{1}{1-\alpha}}. \tag{2.7}$$

After inserting x_1 into (2.6) we obtain the Marshallian demand for the second good

$$x_2^* = D_2(p_1, p_2, M) = \frac{M}{p_2 + p_1\left(\frac{p_1\delta_2}{p_2\delta_1}\right)^{\frac{1}{1-\alpha}}},$$

and from the equation (2.7) then Marshallian demand for the first good

$$x_1^* = D_1(p_1, p_2, M) = \frac{M\left(\frac{p_1\delta_2}{p_2\delta_1}\right)^{\frac{1}{1-\alpha}}}{p_2 + p_1\left(\frac{p_1\delta_2}{p_2\delta_1}\right)^{\frac{1}{1-\alpha}}}.$$

■

Problem 9 *Show that constant elasticity of substitution utility function will transform into (a) linear utility function for $\alpha = 1$, (b) Cobb-Douglas utility function for $\alpha \to 0$ and (c) the Leontief utility function for $\alpha \to -\infty$.*

Solution.

a) After solving for α we get a liner utility function in the following form:

$$u(x_1, x_2) = a_1x_1 + a_2x_2,$$

where $a_1 = A\delta_1$ and $a_2 = A\delta_2$.

b) Let us calculate the limit

$$\begin{aligned}\lim_{\alpha\to 0} A(\delta_1x_1^\alpha + \delta_2x_2^\alpha)^{1/\alpha} &= A\lim_{\alpha\to 0}(\delta_1x_1^\alpha + \delta_2x_2^\alpha)^{1/\alpha}, \\ &= A\lim_{\alpha\to 0} e^{\log(\delta_1x_1^\alpha + \delta_2x_2^\alpha)^{1/\alpha}}.\end{aligned}$$

It is then sufficient to calculate the limit

$$\lim_{\alpha\to 0}\log(\delta_1x_1^\alpha + \delta_2x_2^\alpha)^{1/\alpha} = \lim_{\alpha\to 0}\frac{\log(\delta_1x_1^\alpha + \delta_2x_2^\alpha)}{\alpha}. \tag{2.8}$$

Since it is a limit of type $\frac{0}{0}$, we use the L'Hospital's rule. We get

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\log(\delta_1x_1^\alpha + \delta_2x_2^\alpha)}{\alpha} &= \lim_{\alpha\to 0}\frac{\delta_1x_1^\alpha\log x_1 + \delta_2x_2^\alpha\log x_2}{\delta_1x_1^\alpha + \delta_2x_2^\alpha}, \\ &= \delta_1\log x_1 + \delta_2\log x_2,\end{aligned}$$

so that

$$\begin{aligned}\lim_{\alpha\to 0} A(\delta_1x_1^\alpha + \delta_2x_2^\alpha)^{1/\alpha} &= Ae^{\delta_1\log x_1 + \delta_2\log x_2}, \\ &= Ax_1^{\delta_1}x_2^{\delta_2}.\end{aligned}$$

c) We proceed in the same way as in (b). The limit (2.8) is now of type $\frac{-\infty}{-\infty}$, so that we can again use L'Hospital's rule. Let us further assume that

$\min\{x_1, x_2\} = x_1$. Hence,

$$\begin{aligned}\lim_{\alpha\to-\infty} \frac{\delta_1 x_1^\alpha \log x_1 + \delta_2 x_2^\alpha \log x_2}{\delta_1 x_1^\alpha + \delta_2 x_2^\alpha} &= \lim_{\alpha\to-\infty} \frac{\delta_1 x_1^\alpha \log x_1 + \delta_2 x_2^\alpha \log x_2}{\delta_1 x_1^\alpha + \delta_2 x_2^\alpha} \frac{\frac{1}{x_1^\alpha}}{\frac{1}{x_1^\alpha}}, \\ &= \lim_{\alpha\to-\infty} \frac{\delta_1 \log x_1 + \delta_2 \left(\frac{x_2}{x_1}\right)^\alpha \log x_2}{\delta_1 + \delta_2 \left(\frac{x_2}{x_1}\right)^\alpha}, \\ &= \log x_1.\end{aligned}$$

since $\lim_{\alpha\to-\infty} \left(\frac{x_2}{x_1}\right)^\alpha = 0$. If $\min\{x_1, x_2\} = x_2$, we derive in the same fashion that

$$\lim_{\alpha\to-\infty} \frac{\delta_1 x_1^\alpha \log x_1 + \delta_2 x_2^\alpha \log x_2}{\delta_1 x_1^\alpha + \delta_2 x_2^\alpha} = \log x_2.$$

It then follows that

$$\lim_{\alpha\to-\infty} A(\delta_1 x_1^\alpha + \delta_2 x_2^\alpha)^{1/\alpha} = A \min\{x_1, x_2\}.$$

■

Problem 10 *In his elderly age, Lord Buckingham spends his money only on three goods: precious scriptures for his library, quality wines, and golden jewelry for his young spouse. His indirect utility function has the following form (p_x denoting prices and M income):*

$$v(p_1, p_2, p_3, M) = \frac{3M + 15p_1 + 9p_2}{p_3} + \frac{12p_3}{p_1} + \frac{27p_3}{p_2}$$

a) *Using Roy's identity, derive his Marshallian demand for gold $D_3(p_1, p_2, p_3, M)$.*

b) *Derive his expenditure function $m(p_1, p_2, p_3, u)$.*

c) *Using the expenditure function from the preceding item, derive Hicksian demands for scriptures and wine: $H_1(p_1, p_2, p_3, u)$ and $H_2(p_1, p_2, p_3, u)$.*

d) *The price p_1 was originally £4, p_2 £2 and p_3 £12. Thanks to external circumstances the prices plummeted for p_1 to £3 and for p_3 to £9. However, p_2 increased to £4.5. Income of Lord Buckingham is constant at £2400 per month. Using equivalent variation, measure his benefit or loss from the given price changes.*

Solution.

a) $x_3^* = \dfrac{M + 5p_1 + 3p_2}{p_3} - 4\dfrac{p_3}{p_1} - 9\dfrac{p_3}{p_2}$

b) $m(p_1, p_2, p_3, u) = p_3\left(\dfrac{u}{3} - \dfrac{4p_3}{p_1} - \dfrac{9p_3}{p_2}\right) - 5p_1 - 3p_2$

c) $x_1^* = 4\left(\dfrac{p_3}{p_1}\right)^2 - 5, \quad x_2^* = 9\left(\dfrac{p_3}{p_2}\right)^2 - 3$

d) $EV = 380$

■

Problem 11 *Siao-Tchin, a secretary from Bejing, lives with her boyfriend. Every morning, he drives her to the office, so she spends money only on clothing, cosmetics, and food. The price of clothes p_1^0 is 15 yuan, price of cosmetics p_2^0 10 yuan and price of food p_3^0 5 yuan. Siao-Tchin has a utility function $u(x_1, x_2, x_3) = 3\sqrt{x_1 + 2} + 2\sqrt{x_2 + 1} + \sqrt{x_3}$.*

a) Calculate the daily demanded quantities of all goods x_1^*, x_2^*, x_3^* if her daily income is 50 yuan.

b) Her mother, who lives in the village, got sick and Siao needed to send her a medicine, which costs 30 yuan per day. Therefore, she is left with 20 yuan per day. Given the changed income (prices remained the same), what are now the demanded quantities of goods x_1^*, x_2^*, x_3^*?

Solution.

a)
$$L = 3\sqrt{x_1 + 2} + 2\sqrt{x_2 + 1} + \sqrt{x_3} + \lambda[50 - 15x_1 - 10x_2 - 5x_3]$$
$$\frac{\partial L}{\partial x_1} = \frac{3}{2\sqrt{x_1 + 2}} - 15\lambda \overset{!}{=} 0$$
$$\frac{\partial L}{\partial x_2} = \frac{1}{\sqrt{x_2 + 1}} - 10\lambda \overset{!}{=} 0$$
$$\frac{\partial L}{\partial x_3} = \frac{1}{2\sqrt{x_3}} - 5\lambda \overset{!}{=} 0$$
$$\frac{\partial L}{\partial \lambda} = 50 - 15x_1 - 10x_2 - 5x_3 \overset{!}{=} 0$$
$$\frac{1}{\lambda} = 10\sqrt{x_1 + 2} = 10\sqrt{x_2 + 1} = 10\sqrt{x_3} \implies$$
$$x_2 = x_1 + 1, x_3 = x_1 + 1 + 1 = x_1 + 2$$

After rearranging, we get $x_1 = \underline{\underline{1}},\ x_2 = \underline{\underline{2}},\ x_3 = \underline{\underline{3}}$.

b) $x_2 = x_1 + 1, x_3 = x_1 + 1 + 1 = x_1 + 2$

$20 - 15x_1 - 10x_1 - 10 - 5x_1 - 10 = 0$

$30x_1 = 0 \implies x_1 = \underline{\underline{0}},\ x_2 = \underline{\underline{1}},\ x_3 = \underline{\underline{2}}$

■

Problem 12 *Pamela, a great healthy lifestyle promoter, spends money mostly on going to the gym and her healthy diet. Her utility function has the following form:* $u(x_1, x_2) = 20\sqrt[3]{x_1} \cdot \sqrt[3]{x_2^2} + 3\sqrt[3]{x_3^2}$.
Here, x_1 *is the amount of time spent in the gym,* x_2 *the number of portions of healthy food and supplements and* x_3 *denotes all other goods other than gym and healthy diet. The price of fitness is* $p_1^0 = 120$ *USD, the price of food incl. dietary supplements is* $p_2^0 = 200$ *USD, and the price of the third commodity, representing all other goods, is* $p_3^0 = 1$ *USD. Calculate the optimum number of all goods* x_1^*, x_2^*, x_3^* *within a month if her monthly income is 43 200 USD.*

Solution.

$$L = 1200\sqrt[3]{x_1} \cdot \sqrt[3]{x_2} + x_3 + \lambda[M - p_1x_1 - p_2x_2 - p_3x_3]$$

$$\frac{\partial L}{\partial x_1} = 400\frac{\sqrt[3]{x_2}}{\sqrt{x_1^2}} - \lambda p_1 \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial x_2} = 400\frac{\sqrt[3]{x_1}}{\sqrt{x_2^2}} - \lambda p_2 \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial x_3} = 1 - \lambda p_3 \stackrel{!}{=} 0$$

$$\frac{\partial L}{\partial \lambda} = M - p_1x_1 - p_2x_2 - p_3x_3 \stackrel{!}{=} 0$$

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \implies x_2 = x_1\frac{p_1}{p_2}$$

$$\lambda = \frac{1}{p_3} \implies 400\sqrt[3]{x_1\frac{p_1/p_2}{x_1^2}} = \frac{p_1}{p_3} \implies$$

$$\sqrt[3]{x_1} = 400\sqrt[3]{\frac{p_1/p_2}{p_1^3}} \implies x_1 = \frac{(400p_3)^3}{p_1^2p_2}$$

$$x_2 = \frac{p_1}{p_2}\frac{(400p_3)^3}{p_1^2p_2} = \frac{(400p_3)^3}{p_1p_2^2}$$

$$x_3 = \frac{M - p_1x_1 - p_2x_2}{p_3} = \frac{M}{p_3} - \frac{p_1}{p_3}\frac{(400p_3)^3}{p_1^2p_2} - \frac{p_2}{p_3}\frac{(400p_3)^3}{p_1p_2^2} =$$

$$= \frac{M}{p_3} - 2\frac{400^3p_3^2}{p_1p_2}$$

■

Problem 13 *Hicksian demands of a consumer, who consumes only two goods, are as follows:*

$$H_1(p_1,p_2,u) = \frac{u}{p_1^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^2} - 1, \quad H_2(p_1,p_2,u) = \frac{u}{p_2^2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^2} - 1$$

a) Derive the expenditure function $m(p_1,p_2,u)$ and from it the indirect utility function $v(p_1,p_2,M)$.

b) Out of $v(p_1,p_2,M)$, the Marshallian demand $D_1(p_1,p_2,M)$ derive using Roy's identity.

Solution.

a)
$$m(p,u) = \frac{u}{p_1\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^2} - p_1 + \frac{u}{p_2\left(\frac{1}{p_1}+\frac{1}{p_2}\right)^2} - p_2$$

$$= \frac{u}{\left(\frac{1}{p_1}+\frac{1}{p_2}\right)} - p_1 - p_2$$

$$v(p_1,p_2,M) = (M+p_1+p_2)\left(\frac{1}{p_1}+\frac{1}{p_2}\right)$$

b)
$$\frac{\partial v(p_1,p_2,M)}{\partial p_1} = \left(\frac{1}{p_1}+\frac{1}{p_2}\right) - (M+p_1+p_2)\left(\frac{1}{p_1}\right)^2$$

$$= -(M+p_1+p_2)\cdot\frac{p_2}{p_2p_1^2} + \frac{p_1p_2}{p_2p_1^2} + \frac{p_1^2}{p_2p_1^2} =$$

$$= -\frac{Mp_2+p_2^2-p_1^2}{p_2p_1^2}$$

$$\frac{\partial v(p_1,p_2,M)}{\partial M} = \left(\frac{1}{p_1}+\frac{1}{p_2}\right) = \frac{p_1+p_2}{p_1p_2}$$

$$D_1(p,M) = -\frac{\partial v(p_1,p_2,M)}{\partial p_1}\bigg/\frac{\partial v(p_1,p_2,M)}{\partial M} =$$

$$= \frac{Mp_2+p_2^2-p_1^2}{p_2p_1^2}\bigg/\frac{p_1+p_2}{p_1p_2} = \frac{Mp_2+p_2^2-p_1^2}{(p_1+p_2)p_1}$$

■

Problem 14 *A famous drug baron, Joachim "The Giraffe" Guzmann, left a Mexican prison, and is taking refuge in a neighbouring country, where he has been granted anonymity. After bribing local judges, he was left with a relatively small amount of local currency (1,000,000 pesos) and he cannot expect any additional income. In order to remain hidden there for two periods, he decided to rent and operate houses in the local red-light district. His investment function expressed in thousands of pesos (i.e. $\overline{C_1} = 1000$ {th. pesos}) of real expenditure has the following form:*

$$C_2^0 = 162\sqrt[3]{1000 - C_1^0}$$

Let us assume that there is a high unified discount rate for which it is possible to lend and save money ($r = 200\%$), but also a high inflation rate ($\pi = 100\%$). Joachim will do business only in the first period and will spend all the remaining pesos in the second period.

a) What is the real interest rate ρ in this country?

b) What amount (i.e. $(\overline{C_1} - C_1^0)$ th. pesos) will Joachim invest into his business and what expected income (i.e. C_2^0 th. pesos) it will bring him in the second period?

c) If Joachim's utility function is $u(C) = 3\ln(C_1) + ln(C_2)$, what will be his final real consumption in periods 1 and 2 (C_1^*, C_2^*)?

Solution.

a) $\rho = \frac{r-\pi}{1+\pi} = 0.5$

b) $$-\frac{\partial C_2^0}{\partial C_1^0} = \frac{162}{3}\frac{1}{\sqrt[3]{(1000 - C_1^0)^2}} \overset{!}{=} (1+\rho) = 1.5 \implies$$

$$46656 = 1000^2 - 2000C_1^0 + (C_1^0)^2 \implies$$

$$C_{1\,1/2}^0 = \frac{2000 \pm 432}{2} \implies C_1^0 = \underline{\underline{784}},$$

since the other solution (1216) exceeds the initial endowment.

$$I = \overline{C_1} - C_1^0 = \underline{\underline{216}}$$

$$C_2^0 = 162\sqrt[3]{216} = \underline{\underline{972}}$$

c) $$MRS_{21} = \frac{MU_1}{MU_2} = \frac{3c_2^*}{c_1^*} \stackrel{!}{=} 1 + \rho = 1.5 \implies 2C_2^* = C_1^*$$

$$c_1^* + \frac{c_2^*}{1+\rho} = c_1^0 + \frac{c_2^0}{1+\rho} \implies c_1^* + \frac{c_2^*}{1.5} = 784 + \frac{972}{1.5} \implies$$

$$2C_2^* + \frac{c_2^*}{1.5} = 784 + \frac{972}{1.5} \implies 4C_2^* = 784 \cdot 1.5 + 972 = 2148$$

$$C_2^* = \underline{\underline{537}}, C_1^* = \underline{\underline{1074}}$$

■

Problem 15 *A country has increased its debt by 6 billions local dollars (LD) by selling government bonds. The bonds allow for a regular annual income (coupon) of 100 LD in constant prices annually, from today forever. The country's debt is calculated based on the present value of monetary debt flows.*
The government decided to lower the debt by offering the creditors a participation in a lottery in exchange for the bonds. In this lottery, 20% of tickets win a similar bond, but with an increased annual income (400 LD) in constant prices. The rest of the tickets do not win anything.

a) *Calculate the present value of both types of bonds, if the nominal interest rate is $r = 40\%$ and inflation is $\pi = 20\%$. Let us assume that the closest pay day of the coupon is exactly after one year.*

b) *Paul owns one such '100 LD' bond and his total wealth including the present value of this bond amounts to 2200 LD in constant prices. Would he participate in the lottery or keep his bond, if his utility function for wealth is $u(w) = w^2$? For simplicity, we assume that the lottery takes place already tonight.*

c) *If you found out that he would participate in the lottery, what is the highest amount which he will be willing to pay for the possibility to take part in the lottery?*
Alternatively, if you found out that he would not take part in the lottery, calculate the lowest amount he would accept in exchange for participating in the lottery.

d) *If one quarter of the bonds owned by the creditors would be part of the lottery, how would the government debt change?*

Solution.

a) $$\rho = \frac{r - \pi}{1 + \pi} = \frac{1}{6}$$

$$PV(x^0) = 100 \Big/ \frac{1}{6} = 600, \quad PV(x^1) = 400 \Big/ \frac{1}{6} = 2400$$

b) $EU = 0,2 \cdot (2200 + 2400 - 600)^2 + 0.8 \cdot (2200 - 600)^2$
$= 5248000 > 4840000 = u(w) \implies$ he would participate.

c) $\sqrt{5248000} = 2290.85 > 2200 = w^0 \implies$
he will be willing to pay at least 90 LD.

d) Amount of bonds in the lottery (bil. LD) $= 6 \cdot 0.25 = 1.5$.
Decrease of debt = 1.5 - winners' income =
$1.5 - 0.2 \cdot 1.5 \cdot 4 = 0.3$ bil. LD.

■

Problem 16 *Luke consumes only two goods. His Hicksian demands for these two goods are as follows:*

$$H_1(p_1, p_2, u) = u + 80, \qquad H_2(p_1, p_2, u) = 5u.$$

a) Derive the consumer expenditure function $m(p_1, p_2, u)$ and from that the indirect utility function $v(p_1, p_2, M)$.

b) Derive Marshallian demands $D_1(p_1, p_2, M)$ and $D_2(p_1, p_2, M)$, using Roy's identity.

c) Due to the indirect tax reform, p_1 increased from \$2 to \$5. On the other hand, p_2 decreased from \$6 to \$4. Luke has a stable monthly income of \$800. His sister Leia has the same preferences as him, but as a top-manager of an international company, she has 10 times larger wage, i.e. \$8000. Decide using $v(p_1, p_2, M)$, how this reform influence the utility of Luke and Leia.

d) Calculate the effect of price changes by tracking the compensating variation (CV) and equivalent variation (EV).

Solution.

a) $m(p_1, p_2, u) = p_1(u + 80) + p_2 \cdot 5u = u(p_1 + 5p_2) + 80p_1$

$$v(p_1, p_2, M) = \frac{M - 80p_1}{p_1 + 5p_2}$$

b) $$D_1(p_1, p_2, M) = -\frac{\partial \frac{M-80p_1}{p_1+5p_2} / \partial p_1}{\partial \frac{M-80p_1}{p_1+5p_2} / \partial M} = \frac{\frac{M-80p_1}{(p_1+5p_2)^2} + \frac{80}{p_1+5p_2}}{\frac{1}{p_1+5p_2}} = \frac{M + 400p_2}{p_1 + 5p_2}$$

$$D_2(p_1, p_2, M) = -\frac{\partial \frac{M-80p_1}{p_1+5p_2} / \partial p_2}{\partial \frac{M-80p_1}{p_1+5p_2} / \partial M} = \frac{5M - 400p_1}{p_1 + 5p_2}$$

c) $u^0_{Luke} = \dfrac{M^{Luke} - 80p^0_1}{p^0_1 + 5p^0_2} = \dfrac{800 - 80 \cdot 2}{2 + 5 \cdot 6} = \underline{\underline{20}}$

$u^1_{Luke} = \dfrac{M^{Luke} - 80p^1_1}{p^1_1 + 5p^1_2} = \dfrac{800 - 80 \cdot 5}{5 + 5 \cdot 4} = \underline{\underline{16}}$

$u^0_{Leia} = \dfrac{M^{Leia} - 80p^0_1}{p^0_1 + 5p^0_2} = \dfrac{8000 - 80 \cdot 2}{2 + 5 \cdot 6} = \underline{\underline{245}}$

$u^1_{Leia} = \dfrac{M^{Leia} - 80p^1_1}{p^1_1 + 5p^1_2} = \dfrac{8000 - 80 \cdot 5}{5 + 5 \cdot 4} = \underline{\underline{304}}$

d) $CV^{Luke} = M^{Luke} - m(u^0_{Luke}, p^1_1, p^1_2) \quad = 800 - 20(5 + 5 \cdot 4) - 80 \cdot 5 =$
$= \underline{\underline{-100}}$

$CV^{Leia} = M^{Leia} - m(u^0_{Leia}, p^1_1, p^1_2) \quad = 8000 - 245(5 + 5 \cdot 4) - 80 \cdot 5 =$
$= \underline{\underline{1475}}$

$EV^{Luke} = m(u^1_{Luke}, p^0_1, p^0_2) - M^{Luke} \quad = 16(2 + 5 \cdot 6) + 80 \cdot 2 - 800 =$
$= \underline{\underline{-128}}$

$EV^{Leia} = m(u^1_{Leia}, p^0_1, p^0_2) - M^{Leia} \quad = 304(2 + 5 \cdot 6) + 80 \cdot 2 - 8000 =$
$= \underline{\underline{1888}}$

■

Problem 17 *Nikola is a tram driver in Belgrade and his employer pays him an hourly wage of 120 dinars. He also rents a one-room apartment in his house to tourists for 290 dinars/day. The Belgrade Public Transport Company allows its employees to work for this hourly rate any chosen number of hours, up to the maximum of 12 hours per day. Nikola's utility function for work and consumption is:*

$$u(L, C) = \frac{\sqrt{C + 10}}{L + 10}$$

We assume that no non-working days exist, there is no inflation and Nikola is virtually indifferent about particular hours in the day he will be working. Determine how many hours he will work (L^*) and what will be his real income (C^*).

Solution.

$$C^* = 120L^* + 290, \quad L^* \leq 12$$

$$MRS_{CL} = -\frac{\sqrt{C^*+10}}{(L^*+10)^2} \cdot 2\sqrt{C^*+10} \cdot (L^*+10) = -120 \implies$$

$$\frac{2(C^*+10)}{L^*+10} = 120 \implies 2C^* = 120(L^*+10) - 20 \implies C^* = 60L^* + 590$$

$$60L^* + 590 = 120L^* + 290 \implies 60L^* = 30 \implies L^* = \underline{\underline{5}} \leq 12$$

$$C^* = 120 \cdot 5 + 290 = \underline{\underline{890}}$$

■

Problem 18 *Jaromír Jágr, a famous ice hockey player, decided to think about providing for himself (and his heirs) after the end of his career. If nothing goes wrong, his real annual income will be as follows.*

seasons:	2014/15	2015/16	2016/17	2017/18 onwards
income:	4 mil. USD	4.4 mil. USD	2.42 mil. USD	0 USD

a) *He is thinking about insurance: in case of a serious injury so that he would not be able to play from the first season onward, i.e.* $M_i = 0, i = 2014/15, \ldots, 2016/17$, *this insurance would bring him the same amount as his prospective incomes. Let us assume that a constant amount X would be paid out from the beginning of 2015/16 until infinity. What should this amount X be in case of a fixed discount rate of 10%?*

b) *An insurance company is willing to offer him an insurance with a tariff of* $0.2K$, *where K is the present value of X. Jaromír is risk-neutral. What is the probability of injury this year, so that he would agree to this insurance?*

c) *Let us assume that Miro Šatan has the same income expectations as Jaromír Jágr, and his current wealth amounts to 6 mil. Contrary to Jaromír, he is risk-averse: his* $u(w) = ln(w)$, *and his expected probability of injury in this season is* $\pi = 0.1$. *How large annual amounts X will he chose if X will be collected from the next season until infinity? Let us denote* $K = PV(X)$. *The insurance tariff is the same as Jaromír's, i.e. 0.2 K.*

Solution.

a) $PV\left(M_{14/15}^{16/17}\right) = 4 + 4.4/1.1 + 2.42/1.21 = \underline{\underline{10 \text{ mil. USD}}}$

$$PV(X^\infty) = X/0.1 = 10 \implies X = \underline{\underline{1 \text{ mil. USD}}}$$

b) $\pi > 0.2$

c) $EU = 0.1 \ln\left(6 + (1 - 0.2)\, K\right) + 0.9 \ln(6 + 10 - 0.2K)$

$$\frac{\partial EU}{\partial K} = 0.1 \cdot \frac{0.8}{6 + 0.8K} - 0.9 \cdot \frac{0.2}{16 - 0.2K} \stackrel{!}{=} 0 \implies$$

$$K = 1.25 \text{ mil. USD} = \frac{X}{r} \implies X = \underline{\underline{0.125 \text{ mil. USD}}}$$

■

Problem 19 *A consumer, who consumes only two goods, has the following demand function for the first good:* $D_1(p, M) = \frac{3M - p_1}{3p_1 + 2p_2} > 0$.

a) Derive the expression for the cross elasticity of demand $e^1_{p_2}$*. Determine whether it is a gross substitute or gross complement of the second good. How will expenditure on good 1 react to changes in the price* p_2*?*

b) His income $M = 151$ *and price* $p_2 = \$3$ *remain unchanged, however, the price* p_1 *increased from \$3 to \$6. Calculate the total own effect of this price change on demand for the good 1. Using the compensating variation, decompose this effect into own substitution effect and income effect.*

Solution.

a)
$$\frac{\partial D_1(p, M)}{\partial p_2} = -\frac{2p_2(3M - p_1)}{(3p_1 + 2p_2)^2} \Bigg/ \frac{3M - p_1}{3p_1 + 2p_2} =$$

$$= -\frac{2p_2}{3p_1 + 2p_2} \implies -1 < e^1_{p_2} < 0.$$

$\implies$ gross complement,

expenditures on good 1 decrease with price p_2.

b)
$$D^0_1(p, M) = \frac{3 \cdot 151 - 3}{3 \cdot 3 + 2 \cdot 3} = \underline{\underline{30}}$$

$$D^1_1(p, M) = \frac{3 \cdot 151 - 6}{3 \cdot 6 + 2 \cdot 3} = \underline{\underline{18.625}}$$

$$CV = \Delta p_1(-D^0_1) = 3 \cdot (-30) = \underline{\underline{90}}$$

$$D^I_1(p, M) = \frac{3 \cdot (151 + 90) - 6}{3 \cdot 6 + 2 \cdot 3} = \underline{\underline{29.875}}$$

$$\text{Total effect} = 18.625 - 30 = \underline{\underline{-11.375}}$$

$$\text{Own substitution effect} = 29.875 - 30 = \underline{\underline{-0.125}}$$

$$\text{Income effect} = 18.625 - 29.875 = \underline{\underline{-11.25}}$$

■

Problem 20 *A consumer, who consumes only two goods, has the following demand functions:* $D_1(p, M) = \frac{M-2p_1+10p_2}{2p_1} > 0$, $D_2(p, M) = \frac{M+2p_1-10p_2}{2p_2} > 0$.

a) Derive the expression for the own elasticity of demand $e^2_{p_2}$. *Is the demand for good 2 elastic or inelastic in its own price? How will changes in the price* p_2 *affect expenditure on good 2?*

b) His income $M = 160$ *and price* $p_2 = \$4$ *remain unchanged, however, the price* p_1 *decreased from \$10 to \$5 . Calculate the total cross effect of this price change on demand for the good 2. Using the compensating variation, decompose this effect into the cross substitution effect and income effect.*

Solution.

a)
$$\frac{\partial D_2(p, M)}{\partial p_2} = -\frac{M + 2p_1}{2(p_2)^2} \Bigg/ \frac{M + 2p_1 - 10p_2}{2(p_2)^2} =$$
$$= -\frac{M + 2p_1}{M + 2p_1 - 10p_2} < -1$$
$$D_2 > 0 \implies M + 2p_1 > 10p_2$$
$\implies$ elastic, expenditure will decrease with price p_2.

b)
$$D_1^0(p, M) = \frac{160 - 2 \cdot 10 + 10 \cdot 4}{2 \cdot 10} = 9$$
$$CV = \Delta p_1(-D_1^0) = (-5) \cdot (-9) = 45$$
$$D_2^0(p, M) = \frac{160 + 2 \cdot 10 - 10 \cdot 4}{2 \cdot 4} = 17.5$$
$$D_2^1(p, M) = \frac{160 + 2 \cdot 5 - 10 \cdot 4}{2 \cdot 4} = 16.25$$
$$D_2^I(p, M) = \frac{160 - 45 + 2 \cdot 5 - 10 \cdot 4}{2 \cdot 4} = 10.625$$
$$\text{Total effect} = 16.25 - 17.5 = \underline{\underline{-1.25}}$$
$$\text{Cross substitution effect} = 10.625 - 17.5 = \underline{\underline{-6.875}}$$
$$\text{Income effect} = 16.25 - 10.625 = \underline{\underline{5.625}}$$

■

Chapter 3

Theory of the Firm and Market Structures

Problem 21 *A perfectly competitive firm has the following cost function:* $C(w,y) = \left(\frac{y+9}{60}\right)^2 \cdot \frac{w_1 w_2}{w_1+100w_2} - 81 \cdot w_2$.

a) *What are its $MC(w,y)$ and $AC(w,y)$?*

b) *Determine the supply function $y^* = y(w_1, w_2, p)$ for generally set prices w_1, w_2, p, where p is greater than p_0.*

c) *Determine the profit function $\pi(w_1, w_2, p)$.*

d) *Verify the validity of Hotelling's lemma for the relationship between $\pi(w,p)$ and $y(w,p)$.*

e) *Derive Marshallian demands $z_1(w,p)$ and $z_2(w,p)$.*

Solution.

a) $$MC(w,y) = \frac{\partial C(w,y)}{\partial y} = 2 \cdot \frac{y+9}{60^2} \frac{w_1 w_2}{w_1 + 100w_2}$$

$$AC(w,y) = \frac{C(w,y)}{y} = \left(\left(\frac{y+9}{60}\right)^2 \cdot \frac{w_1 w_2}{w_1 + 100w_2} - 81 \cdot w_2\right) \Big/ y$$

b) $$2\left(\frac{y+9}{60^2}\right) \frac{w_1 w_2}{w_1 + 100w_2} = p \implies$$

$$y(p,w) = \frac{3600p}{2} \cdot \frac{w_1 + 100w_2}{w_1 w_2} - 9 = 1800p \cdot \frac{w_1 + 100w_2}{w_1 w_2} - 9$$

c) $$\pi(w_1, w_2, p) = 1800p^2 \cdot \frac{w_1 + 100w_2}{w_1 w_2} - 9p - \left(\frac{1800}{60} \cdot p \cdot \frac{w_1 + 100w_2}{w_1 w_2}\right)^2 \frac{w_1 w_2}{w_1 + 100w_2} + 81w_2$$
$$= 900p^2 \cdot \frac{w_1 + 100w_2}{w_1 w_2} - 9p + 81w_2$$

d) $$\frac{\partial \pi(w_1, w_2, p)}{\partial p} = 1800p \cdot \frac{w_1 + 100w_2}{w_1 w_2} - 9$$

e) $$\frac{\partial \pi(w_1, w_2, p)}{\partial w_1} = 900p^2 \cdot \left(-\frac{100}{w_1^2}\right) = -\left(\frac{300p}{w_1}\right)^2$$
$$z_1(w_1, w_2, p) = \left(\frac{300p}{w_1}\right)^2$$
$$\frac{\partial \pi(w_1, w_2, p)}{\partial w_2} = 900p^2 \cdot \left(-\frac{1}{w_2^2}\right) + 81 = -\left(\frac{30p}{w_2}\right)^2 + 81$$
$$z_2(w_1, w_2, p) = \left(\frac{30p}{w_2}\right)^2 - 81$$

■

Problem 22 *A company has the following production function:* $y = f(z_1, z_2, z_3) = 2\sqrt{z_1 + 15z_2} + z_3$, *where* z_1 *represents the number of hours worked,* z_2 *the number of engine-hours and* z_3 *the amount of material. The company has 60 engine-hours at its disposal in the short term (1 month), which it cannot either increase or resell* ($\overline{z_2}$)*. It already paid for each of the hours for this time period* $w_2 = 4500$*.*

a) Derive the short-term production function $f(z_1, 60, z_3)$

b) For positive amounts of all inputs determine conditional short-term demand functions for labour z_1 *and material* z_3*.*

c) Derive the corresponding short-term cost function $c(w, y, 60)$ *and verify the validity of Shephard's lemma for both prices of variable inputs* w_1 *and* w_3*.*

Solution.

a) $y = f(z_1, z_2, z_3) = 2\sqrt{z_1 + 900} + z_3$

b) E.g. $$MRTS_{21} = \frac{1}{\sqrt{z_1 + 900}} = \frac{w_1}{w_3} \implies z_1(y, w) = \left(\frac{w_3}{w_1}\right)^2 - 900.$$

After substituting $z_1(y, w)$ for z_1 in the production function, we get

$$z_3 = y - 2\frac{w_3}{w_1}.$$

c)
$$\begin{aligned} c(w,y) &= w_1 z_1(y,w) + w_3 z_3(y,w) + 4500 \cdot 60 \\ &= w_1\left(\left(\frac{w_3}{w_1}\right)^2 - 900\right) + w_3\left(y - 2\frac{w_3}{w_1}\right) + 270000 \\ &= w_3 y - \frac{w_3^2}{w_1} - 900w_1 + 270000 \\ \frac{\partial c(w,y)}{\partial w_1} &= \left(\frac{w_3}{w_1}\right)^2 - 900 = z_1(y,w) \\ \frac{\partial c(w,y)}{\partial w_3} &= y - 2\frac{w_3}{w_1} = z_3(y,w) \end{aligned}$$

■

Problem 23 *There is a new flight route from Madrid to Mexico City. The current flight company operates two flights, at 9 AM and at 9 PM. One ticket costs 1000 EUR. A new low-cost company is considering joining this market with one flight at 3 PM. Let us assume that his MC=AC=200. The number of people willing to fly on this route between 9 AM and 9 PM is 120 (i.e. 10 per hour(!)) and all would fit inside one plane. These people have perfectly inelastic demand for the flight from Madrid to Mexico City (they will surely fly but it does not matter with which company). They are evenly distributed in time (i.e. their preferred departure times) and their subjective cost from adapting to a flight schedule is 100 EUR/hour.*

a) What is the price of the new low-cost company, in order to maximize its profit?

b) If the cost of launching a single airplane costs 40 000 EUR, irrespective of the number of passengers, is it worth joining the market?

Solution.

a) $p + 100d_i = 1000 + 100(6 - d_i) \implies d = 2d_i = 16 - 0.01p$

$$\begin{aligned} \Pi &= \frac{Q}{h} \cdot (16 - 0.01p) \cdot (p - 200) - FC \\ \frac{\partial \Pi}{\partial p} &= \frac{Q}{h}(16 - 0.02p^* + 2) \stackrel{!}{=} 0 \\ p^* &= \frac{18}{0.02} = \underline{\underline{900}} \end{aligned}$$

b) $d = 16 - 0.01p^* = 16 - 0.01 \cdot 900 = \underline{\underline{7 \text{ hours}}}$

$q^* = 7 \cdot 10 = \underline{\underline{70}}$

$\Pi = 70 \cdot (900 - 200) - 30000 = 49000 - 40000 = \underline{\underline{9000}}$

■

Problem 24 *Al Bunda has set up a gentlemen's club 'No Madam', where only his old friends and their sons are allowed to enter. The club is meant for drinking beer and for slandering women. Whereas slandering and gossiping are free, the beer is charged for. All the friends have the following demand:* $D_s(p) = 6 - \frac{p}{2}$*. Their sons, and Al's son Bud, have the following demand:* $D_j(p) = 6 - p$*. Al's costs for each bottle of beer is \$2 . What will be the price of one beer and the entrance fee to the club (i.e., Al's garage) for seniors and juniors if Al applies perfect price discrimination with two-part tariff?*

Solution.

$$p_s = p_j = MC = 2$$

$$p_s = 12 - 2q_s = 2 = MC \implies q_s = \frac{10}{2} = 5$$

$$p_j = 6 - q_j = 2 = MC \implies q_j = 4$$

entrance fees:

$$F_s = \frac{5 \cdot (12 - 2)}{2} = 25$$

$$F_j = \frac{4 \cdot (6 - 2)}{2} = 8$$

■

Problem 25 *A local isolated ski lift operator, which at the given place constitutes a monopoly, has total costs per hour of operation:* $TC(Q) = 2Q^2 + 7Q + 357$*. His services are demanded by two consumer segments: tourists, whose demand function is* $D(p_1) = \frac{103 - p_1}{2}$ *consumers per hour, and locals with demand function* $D(p_2) = \frac{75 - p_2}{4}$ *consumers per hour.*

a) What is the optimum number of skiers per hour in individual segments, in case the company applies third-degree price discrimination? What will be the respective prices per hour and the company's profit per hour?

b) Due to the EU regulation, the company needs to entertain a single price. What will now be the optimum number of skiers per hour, price per hour of skiing and company's profit?

c) *Count separately the number of local skiers and tourists per hour using the new price. How the new regulation affects the well-being of consumers in both segments?*

Solution.

a)
$$\Pi = (103 - 2q_1)q_1 + (75 - 4q_2)q_2 - 2(q_1 + q_2)^2 - 7(q_1 + q_2) - 357$$
$$\frac{\partial \Pi}{\partial q_1} = 103 - 4q_1 - 4q_1 - 4q_2 - 7 \overset{!}{=} 0 \implies 4q_1 = 48 - 2q_2$$
$$\frac{\partial \Pi}{\partial q_2} = 75 - 8q_2 - 4q_1 - 4q_2 - 7 \overset{!}{=} 0$$
$$\implies 75 - 8q_2 - 48 + 2q_2 - 4q_2 - 7 = 0 \implies q_2 = \underline{\underline{2}} \implies q_1 = \underline{\underline{11}}$$
$$p_1 = 103 - 2q_1 = \underline{\underline{81}}, \quad p_2 = 75 - 4q_2 = \underline{\underline{67}} \implies \Pi = \underline{\underline{239}}$$

b)
$$D(p) = D(p_1) = \frac{103 - p}{2} \quad \text{for} \quad p > 75$$
$$D(p) = D(p_1) + D(p_2) = \frac{206 - 2p + 75 - p}{4} =$$
$$= \frac{281 - 3p}{4} \quad \text{for} \quad p \leq 75$$

For $p > 75$:
$$\Pi = (103 - 2Q)Q - 2Q^2 - 7Q - 357$$
$$\frac{\partial \Pi}{\partial Q} = 103 - 4Q - 4Q - 7 \overset{!}{=} 0$$
$$8Q = 96 \implies Q = 12 \implies p = 103 - 24 = 79 \implies$$
$$\Pi = 79 \cdot 12 - 2 \cdot 144 - 7 \cdot 12 - 357 = 219$$

For $p \leq 75$:
$$\Pi = \left(\frac{281}{3} - \frac{4}{3}Q\right) Q - 2Q^2 - 7Q - 357$$
$$\frac{\partial \Pi}{\partial Q} = \frac{281}{3} - \frac{8}{3}Q - 4Q - \frac{21}{3} \overset{!}{=} 0$$
$$\frac{20}{3}Q = \frac{260}{3} \implies Q = 13 \implies$$
$$p = \frac{281 - 4 \cdot 13}{3} = 76.\overline{3} \implies$$
$$\Pi = 76.\overline{3} \cdot 13 - 2 \cdot 169 - 7 \cdot 13 - 357 = 206.\overline{3} < 219 \implies$$

This is not a maximum of Π, because $p = 76.\overline{3} > 75$, therefore $q_2 = 0$.

c) $q_2 = 0, q_1 = 12$, tourists will be better off:

$$\frac{(103-79)\,12-(103-81)\,11}{2} = 46$$

All utility from exchange for local skiers will dissipate:

$$0 - \frac{(103-67)\,2}{2} = -36$$

■

Problem 26 *Let us assume a duopoly of price competitors with a diversified product. Their cost functions are $TC_1(q_1) = 4q_1 + 5000$ and $TC_2(q_2) = 4q_2 + 5000$. Their demand functions are $D_1(p_1, p_2) = 160 - 2p_1 + p_2$ and $D_2(p_1, p_2) = 160 - 2p_2 + p_1$.*

a) Find Bertrand equilibrium p_1^b, p_2^b, the corresponding outputs q_1^b, q_2^b and profits Π_1^b, Π_2^b.

b) Firms agreed upon unfair competition and set up collusive prices p_1^k, p_2^k. What will be their values, and the corresponding outputs q_1^k, q_2^k and profits Π_1^k, Π_2^k?

c) What is the regenade price p_1^{rp} of the first company in case it decides to defect (i.e. to break the price agreement) and so to maximize its profit? What will be its output q_1^{rp} and profit Π_1^{rp}? What will be the output and profit of the dutiful second company q_2 and Π_2?

d) If the first company assumes that in any other round the collusion may end with a 40% probability due to reasons which it cannot influence and if its discount rate is and remains at 10%, is it profitable for this company to remain renegade?

Solution.

a)
$$\Pi_1(p_1, p_2) = (160 - 2p_1 + p_2)(p_1 - 4) - 5000,$$
$$\frac{\partial \Pi_1(p_1, p_2)}{\partial p_1} = 160 - 4p_1 + p_2 + 8 \stackrel{!}{=} 0,$$
$$168 + p_2 = 4p_1 \wedge p_1 = p_2 \implies p_1^B = p_2^B = 168/3 = \underline{\underline{56}}.$$

This result can be as well derived from

$$\Pi_2(p_1, p_2) = (160 - 2p_2 + p_1)(p_2 - 4) - 5000,$$
$$\frac{\partial \Pi_2(p_1, p_2)}{\partial p_2} = 160 - 4p_2 + p_1 + 8 \stackrel{!}{=} 0.$$

Finally,

$$q_1^B = q_2^B = 160 - 56 = \underline{\underline{104}},$$
$$\Pi_1^B = \Pi_2^B = (56-4)104 - 5000 = \underline{\underline{408}}.$$

b) $$(\Pi_1 + \Pi_2)(p_1, p_2) = (160 - 2p_1 + p_2)(p_1 - 4) + \\ + (160 - 2p_2 + p_1)(p_2 - 4) - 10000,$$
$$\frac{\partial(\Pi_1 + \Pi_2)(p_1, p_2)}{\partial p_1} = 160 - 4p_1 + p_2 + 8 + p_2 - 4 \overset{!}{=} 0,$$
$$82 + p_2 = 2p_1 \wedge p_1 = p_2 \implies p_1^k = p_2^k = \underline{\underline{82}}.$$

This result can be as well derived from

$$\frac{\partial(\Pi_1 + \Pi_2)(p_1, p_2)}{\partial p_2} = 160 - 4p_2 + p_1 + 8 + p_1 - 4 \overset{!}{=} 0.$$

Finally,

$$q_1^k = q_2^k = 160 - 82 = \underline{\underline{78}},$$
$$\Pi_1^k = \Pi_2^k = (82-4)78 - 5000 = \underline{\underline{1084}}.$$

c) $$\frac{\partial\Pi_1(p_1, p_2)}{\partial p_1} = 160 - 4p_1 + p_2 + 8 \overset{!}{=} 0 \implies p_1^{rp} = 42 + \frac{p_2^k}{4} = \frac{82}{4} = \underline{\underline{62.5}},$$
$$q_1^{rp} = 160 - 125 + 82 = \underline{\underline{117}},$$
$$\Pi_1^{rp} = (62.5 - 4)117 - 5000 = \underline{\underline{1844.5}}.$$

The dutiful company keeps the same price as calculated in the case for collusion in b). Therefore, from its demand function,

$$q_2 = 160 - 164 + 62.5 = \underline{\underline{\frac{117}{2}}},$$
$$\Pi_2 = (82 - 4)\frac{117}{2} - 5000 = \underline{\underline{-437}}.$$

d) $$\frac{r + \pi}{1 - \pi} = \frac{5}{6} \leq \frac{8}{9} = \frac{\Pi_1^k - \Pi_1^b}{\Pi_1^{rp} - \Pi_1^k} \implies$$

It is not profitable for the company to remain renegade.

■

Problem 27 *Catherine Z., a famous promoter of bio-technologies and sustainable energy, decided to apply her knowledge in her own enterprise in Brasil's Minas Gerais region, known for its sugarcane production: two minor power* *plants processing waste products of sugarcane. These power plants provide electricity for the whole local closed electric power grid. The first power plant has the cost function* $STC_1(y_1) = 2y_1^3 - 30y_1^2 + 300y_1 + FC_1$*, and the second plant has the following cost function:* $STC_2(y_2) = 2y_2^3 - 20y_2^2 + 250y_2 + FC_2$*. We measure hourly costs in BRL (Brazilian real) and output in MWh.*

a) *Catherine supposes that the total demand would be* $y_0 = 5$*. How should she distribute total output between her power plants so that total costs would be minimized?*

b) *It is estimated that local economic growth will double the demand for electricity, i.e. to* $y_1 = 10$*. How should Catherine proceed in this case in order to minimize total costs?*

Solution.

a) If both power plants would be running, optimal distribution of outputs will be reached in case $SMC_1 = SMC_2$.

$$y_2 = y^0 - y_1 = 5 - y_1$$
$$SMC_1 = 6y_1^2 - 60y_1 + 300 = 6(5 - y_1)^2 - 40(5 - y_1) + 250 = SMC_2$$

After some adjustment, we get

$$y_1 = \underline{\underline{2.5}},$$
$$y_2 = y^0 - y_1 = 5 - 2.5 = \underline{\underline{2.5}}.$$

Then, it is important to verify whether subadditivity does not hold in this case, i.e. if only one power plant should produce all output, and if so, which one.

$$\begin{aligned} VC(2.5, 2.5) &= 2(2.5)^3 - 30(2.5)^2 + 300 \cdot 2,5 + \\ &\quad + 2(2,5)^3 - 20(2.5)^2 + 250 \cdot 2.5 = 1125 \\ VC(5, 0) &= 2(5)^3 - 30(5)^2 + 300 \cdot 5 = \underline{1000} < 1125 \\ VC(0, 5) &= 2(5)^3 - 20(5)^2 + 250 \cdot 5 = \underline{1000} < 1125 \end{aligned}$$

Whether she chooses any of the power plants, all the output should be produced only in one of them. If she estimates that the output will be more than 5 MWh, then, based on the STC, it should be produced in the first one. If she assumes that the output will be less than 5 MWh, then, again based on the STC, in the second one.

b) The procedure remains the same, but conclusions differ.

$$y_2 = y^0 - y_1 = 10 - y_1$$
$$SMC_1 = 6y_1^2 - 60y_1 + 300 = 6(10 - y_1)^2 - 40(10 - y_1) + 250 = SMC_2$$

After some adjustment, we get

$$y_1 = \underline{\underline{7.5}},$$
$$y_2 = y^0 - y_1 = 10 - 7.5 = \underline{\underline{2.5}}.$$

Again, we verify whether there is subadditivity and if only plant (and which one) should produce the output.

$$\begin{aligned} VC(7.5, 2.5) &= 2(7.5)^3 - 30(7.5)^2 + 300 \cdot 7,5 + \\ &\quad + 2(2,5)^3 - 20(2.5)^2 + 250 \cdot 2.5 = 1937.5 \\ VC(10, 0) &= 2(10)^3 - 30(10)^2 + 300 \cdot 10 = \underline{2000} > 1937.5 \\ VC(0, 10) &= 2(10)^3 - 20(10)^2 + 250 \cdot 10 = \underline{2500} > 1937.5 \end{aligned}$$

It is obvious that the output should be produced in both of them: 7.5 MWh and 2.5 MWh, respectively.

■

Problem 28 *Let us assume a duopoly of price competitors with a diversified product, which have the following cost functions:* $TC_1 = 4q_1 + 5000$ *and* $TC_2 = 4q_2 + 5000$. *The respective demand functions are* $D_1(p_1, p_2) = 160 - 2p_1 + p_2$ *and* $D_2(p_1, p_2) = 160 - 2p_2 + p_1$.

a) Find Bertrand equilibrium p_1^b, p_2^b, *the corresponding outputs* q_1^b, q_2^b *and profits* Π_1^b, Π_2^b.

b) Companies agreed on unfair competition by means of setting collusion prices p_1^c, p_2^c. *Determine them, together with the respective outputs* q_1^c, q_2^c *and profits* Π_1^c, Π_2^c.

c) If the first company decides to defect (i.e. to violate the price agreement) and maximize its own profit, what renegade price p_1^{rp} *does it choose? What will be the resulting output* q_1^{rp} *and profit* Π_1^{rp}*?*

d) If the first company assumes that in any other round the collusion can end with a probability of 40% (due to reasons out of its reach), and if its discount rate is and remains at 10%, does it pay off for it to be a renegade?

Solution.

a)
$$\begin{aligned}
\Pi_1(p_1,p_2) &= (160-2p_1+p_2)(p_1-4)-5000\\
\Pi_2(p_1,p_2) &= (160-2p_2+p_1)(p_2-4)-5000\\
\frac{\partial \Pi_1(p_1,p_2)}{\partial p_1} &= 160-4p_1+p_2+8 \overset{!}{=} 0\\
\frac{\partial \Pi_2(p_1,p_2)}{\partial p_2} &= 160-4p_2+p_1+8 \overset{!}{=} 0\\
168+p_2 = 4p_1 \wedge p_1 = p_2 &\implies p_1^b = p_2^b = \underline{\underline{56}}\\
q_1^b = q_2^b &= 160-56 = \underline{\underline{104}}\\
\Pi_1^b = \Pi_2^b &= (56-4)\cdot 104 - 5000 = \underline{\underline{408}}
\end{aligned}$$

b)
$$\begin{aligned}
(\Pi_1+\Pi_2)(p_1,p_2) &= (160-2p_1+p_2)(p_1-4)+\\
&\quad + (160-2p_2+p_1)(p_2-4)-10000\\
\frac{\partial (\Pi_1+\Pi_2)(p_1,p_2)}{\partial p_1} &= 160-4p_1+p_2+8+p_2-4 \overset{!}{=} 0\\
\frac{\partial (\Pi_1+\Pi_2)(p_1,p_2)}{\partial p_2} &= 160-4p_2+p_1+8+p_1-4 \overset{!}{=} 0\\
82+p_2 = 2p_1 \wedge p_1 = p_2 &\implies p_1^c = p_2^c = \underline{\underline{82}}\\
q_1^c = q_2^c &= 160-82 = \underline{\underline{78}}\\
\Pi_1^c = \Pi_2^c &= (82-4)78-5000 = \underline{\underline{1084}}
\end{aligned}$$

c)
$$\begin{aligned}
\frac{\partial \Pi_1\,(p_1,p_2)}{\partial p_1} &= 160-4p_1+p_2+8 \overset{!}{=} 0 \implies\\
p_1^{rp} &= 42+\frac{p_2^c}{4} = \underline{\underline{62.5}}\\
q_1^{rp} &= 160-125+82 = \underline{\underline{117}}\\
\Pi_1^{rp} &= (62.5-4)117-5000 = \underline{\underline{1844.5}}
\end{aligned}$$

d)
$$\frac{r+\pi}{1-\pi} = \frac{5}{6} \le \frac{8}{9} = \frac{\Pi_1^c-\Pi_1^b}{\Pi_1^{rp}-\Pi_1^c} \implies$$

It does not pay off to defect from a collusion.

■

Problem 29 *Let us assume a duopoly model with homogeneous product, where the demand function is* $D_Q(p) = 90 - \frac{p}{2}$. *Firms have the following cost functions:* $C_1(q_1) = 2q_1^2 + 20q_1 + 60$, $C_2(q_2) = 3q_2^2 + 12q_2 + 24$.

a) *The firms can agree to an unfair competition, so that they would pledge to produce outputs* q_1^{uc}, q_2^{uc}, *thereby maximizing the common profit* $\Pi_1^{uc} + \Pi_2^{uc}$ *(i.e. if they agree in the beginning, it is not worth for anybody to cancel the agreement). Determine* q_1^{uc}, q_2^{uc}, *the price* p^{uc} *and profits of unfair competitors* Π_1^{uc}, Π_2^{uc}.

b) *Let us assume that the situation has changed and it is worth for the producer 2 to break the agreement. The producer 1 is not aware of this and therefore continues to produce* q_1^{uc}. *Determine the renegade output of the second producer* q_2^r *under the assumption that he maximizes his profit by choosing the optimum output, not price. For simplicity, round* q_2^r *to the nearest integer and calculate the price* p^r *and profits of both producers. What are the resulting profits for both companies?*

Solution.

a)
$$
\begin{aligned}
D_Q(p) &= 90 - \frac{p}{2} \implies p(q_1, q_2) = 180 - 2q_1 - 2q_2 \\
\Pi_1(q_1, q_2) + \Pi_2(q_1, q_2) &= \left[(180 - 2q_1 - 2q_2)\, q_1 - 2q_1^2 - 20q_1 - 60\right] + \\
&\quad + \left[(180 - 2q_1 - 2q_2)\, q_2 - 3q_2^2 - 12q_2 - 24\right] \\
\frac{\partial \Pi_1(q_1, q_2) + \Pi_2(q_1, q_2)}{\partial q_1} &= 180 - 4q_1 - 2q_2 - 4q_1 - 20 - 2q_2 \stackrel{!}{=} 0 \\
8q_1 &= 160 - 4q_2 \implies q_1 = 20 - \frac{q_2}{2} \\
\frac{\partial \Pi_1(q_1, q_2) + \Pi_2(q_1, q_2)}{\partial q_2} &= -2q_1 + 180 - 2q_1 - 4q_2 - 6q_2 - 12 \stackrel{!}{=} 0 \\
10q_2 &= 168 - 4q_1 \implies \\
10q_2 &= 168 - 80 - 2q_2 \implies q_2^{uc} = \underline{\underline{11}} \\
q_1^{uc} &= 20 - \frac{11}{2} = \underline{\underline{14.5}}, \quad p^{uc} = 180 - 29 - 22 = \underline{\underline{129}} \\
\Pi_1^{uc} &= p^{uc} \cdot q_1^{uc} - C(q_1^{uc}) = \\
&= 129 \cdot 14.5 - 2 \cdot 14.5^2 - 20 \cdot 14.5 - 60 = \underline{\underline{1100}} \\
\Pi_2^{uc} &= p^{uc} \cdot q_2^{uc} - C(q_2^{uc}) = \\
&= 129 \cdot 11 - 3 \cdot 11^2 - 12 \cdot 11 - 24 = \underline{\underline{900}}
\end{aligned}
$$

b)
$$\Pi_2(q_1^{uc}, q_2) = (180 - 2q_1^{uc} - 2q_2)q_2 - 3q_2^2 - 12q_2 - 24 =$$
$$= (180 - 29 - 2q_2)q_2 - 3q_2^2 - 12q_2 - 24$$
$$\frac{\partial \Pi_2(q_1^{uc}, q_2)}{\partial q_2} = 180 - 29 - 4q_2 - 6q_2 - 1 \stackrel{!}{=} 0$$
$$10q_2 = 180 - 41 = 139 \implies q_2^r = 13.9 \approx \underline{\underline{14}} \implies$$
$$p^r = 180 - 29 - 28 = \underline{\underline{123}}$$
$$\Pi_1^{uc} = p^r \cdot q_1^{uc} - C(q_1^{uc}) =$$
$$= 123 \cdot 14.5 - 2 \cdot 14.5^2 - 20 \cdot 14.5 - 60 = \underline{\underline{1013}}$$
$$\Pi_2^r = p^r \cdot q_2^r - C(q_2^r) = 123 \cdot 14 - 3 \cdot 14^2 - 12 \cdot 14 - 24 = \underline{\underline{942}}$$

■

Problem 30 *Imagine a village alongside an 8 kilometer long road in the east-west direction, whose representatives are thinking of building a swimming pool 2 km from the east end (and therefore 6 km from the west end). In the vicinity of the village, there are two natural swimming areas: one close the east end and the second close the west end. The operator in the east is located 2 km from the end of the village, and charges $p_e = \$60$ per day, the operator in the west is 4 km from the end of the village and charges $p_w = \$50$. All those interested in swimming are situated inside the village, i.e. none of them is within the aforementioned 2 or 4 km margin. Furthermore, these prospective consumers are uniformly distributed in space within the village and have perfectly inelastic demand. They also decide only which swimming pool to attend, and not whether they are going to swim or not. The subjectively perceived cost for the journey to the swimming pool is \$5/km, in both directions – there and back.*

a) *Determine the price for all-day ticket which maximizes the profit of the new swimming pool. The expected daily costs per customer are MC=AVC=30.*

b) *Determine the servicing area of the new swimming pool in the east and west direction (d_e and d_w).*

c) *What is the minimum amount of customers per year given the current prices (incl. those who choose other swimming pools), so that it still pays off for the village to build the swimming pool, if yearly fixed cost for the maintenance of the new swimming pool are 300 000 \$?*

Solution.

a)
$$
\begin{aligned}
p_0 + 2td_e &= p_e + 2 \cdot 2t + 2t(2 - d_e) \\
p_0 + 10d_e &= 60 + 20 + 10(2 - d_e) \\
d_e &= 5 - \frac{p_0}{20} \\
p_0 + 2td_w &= p_w + 4 \cdot 2t + 2t(6 - d_w) \\
p_0 + 10d_w &= 50 + 40 + 10(6 - d_w) \\
d_w &= 7.5 - \frac{p_0}{20} \implies d = d_e + d_w = 12.5 - \frac{p_0}{10} \\
q_0(p_0) &= \frac{Q}{\text{km}} \left(12.5 - \frac{p_0}{10}\right) = \frac{Q}{8} \left(12.5 - \frac{p_0}{10}\right) \\
\Pi(p) &= \frac{Q}{\text{km}} (p_0 - 30) \left(12.5 - \frac{p_0}{10}\right) - FC \\
\frac{\partial \Pi(p)}{\partial p_0} &= \frac{Q}{\text{km}} \left(12.5 - \frac{p_0}{3} + 3\right) \stackrel{!}{=} 0 \implies p_0 = 15.5 \cdot 5 = \underline{\underline{\underline{77.5}}}
\end{aligned}
$$

b)
$$
\begin{aligned}
d_e &= 5 - \frac{p_0}{20} = 5 - \frac{77.5}{20} = \underline{\underline{\underline{1.125}}} \\
d_w &= 7.5 - \frac{p_0}{20} = 7.5 - \frac{77.5}{20} = \underline{\underline{\underline{3.625}}}
\end{aligned}
$$

c)
$$
\begin{aligned}
(77.5 - 30) \cdot Q \cdot \frac{4.75}{8} &= 300000 \\
Q &\approx \underline{\underline{\underline{10637}}}
\end{aligned}
$$

■